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Logical Foundation of Quantum Mechanics

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The subject of this article is the reconstruction of quantum mechanics on the basis of a formal language of quantum mechanical propositions. During recent years, research in the foundations of the language of science has given rise to a *dialogic semantics* that is adequate in the case of a formal language for quantum physics. The system of *sequential logic* which is comprised by the language is more general than classical logic; it includes the classical system as a special case. Although the system of sequential logic can be founded without reference to the empirical content of quantum physical propositions, it establishes an essential part of the structure of the mathematical formalism used in quantum mechanics. It is the purpose of this paper to demonstrate the connection between the formal language of quantum physics and its representation by mathematical structures in a self-contained way.

INTRODUCTION

This paper is concerned with the logical structure of propositions in the object language of quantum physics. Its purpose is to show how this structure can be founded by means of a proof theoretic semantics of the language of quantum physics.

The approach considered here has its roots in an article by Birkhoff and v. Neumann (1936) entitled "The Logic of Quantum Mechanics." Since then many authors have investigated possibilities of (re)constructing quantum theory on the basic concept of a quantum mechanical proposition and by means of the particular properties of these propositions. A most elaborate axiomatic approach of this kind is presented by Jauch (1968) and Piron (1976). It is assumed that quantum mechanical propositions form a complete orthocomplemented quasimodular lattice which, in addition, satisfies the atomicity and covering property. This structure, also

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called a *propositional system*, is realized by the closed subspaces of a Hilbert space or by the projection operators defined on a Hilbert space. The propositional system has many similarities with the system of classical propositional logic which can be represented by a Boolean lattice.

The foundation of quantum physics on the particular properties of quantum mechanical propositions has led to the further question, much discussed, whether at least a part of the quantum mechanical propositional system can be understood as a *logical system*, often called *quantum logic*. This is precisely the subject of a logical foundation of quantum mechanics considered here.

An extension of the system of intuitionistic and classical logic to the system of quantum logic from the point of view of an operational foundation of logic is investigated by Mittelstaedt (1978) and his collaborators. By means of idealizing assumptions concerning proofs of *elementary propositions* and by means of *dialogic* proof procedures for *logically connected propositions* a formal language is established that is adequately represented by the system of quantum logic. This system forms a free orthocomplemented quasimodular lattice. In this way an essential part of the quantum mechanical propositional system can indeed be understood as a logical system.

The language of quantum physics requires still another kind of proposition, namely, propositions that are not logically composed of elementary propositions but are *sequentially* composed of elementary propositions. Such *sequential propositions* concern the *evolution* of a quantum mechanical system due to the Hamiltonian of the system as well as due to a sequence of measuring processes with respect to the system. It is the purpose of this paper to extend the concept of a quantum mechanical proposition to sequential propositions and to extend the system of quantum logic to the system of *sequential quantum logic*.

In Section 1 of the paper it is shown how sequential propositions can be defined systematically by means of dialogic proof procedures. In this way a unitary *proof theoretic semantics* is established for logically and sequentially connected propositions. In the framework of this approach one can assign a fundamental role to the sequential connectives. One reobtains logically connected quantum mechanical propositions on the basis of sequential propositions by means of an additional fundamental proposition which is called a *commensurability proposition* since it guarantees the commensurability of the subpropositions.

In Section 2 of the paper the *material logic* and the *formal logic* of quantum mechanical propositions are considered. The material logic consists of an algorithm which allows one to deduce *materially true* compound propositions if a sequence of proofs and disproofs of elementary and commensurability propositions, which are also called *material propositions*, is initially given. It is a logical system which still includes a "semantical" rule due to the initial proofs and disproofs of material propositions. On the other hand, the formal logic does not depend on the specific interpretation and, hence, on the contingent proofs and disproofs of material propositions. The calculus of formal logic establishes *formally true* propositions. It is only mentioned here that the calculi of material logic and of formal logic are *complete* and *sound* with respect to the proof theoretic semantics of the language of quantum mechanical propositions.

In Section 3 of the paper we finally proceed to *algebraic representations* of the formal logic and to *realizations* within the Hilbert space formalism of quantum mechanics. An algebraic representation of the logical system is obtained by constructing the Lindenbaum–Tarski algebra of the sequential quantum logic. This Lindenbaum–Tarski algebra possesses the well-known structure of a Baer* semigroup (Foulis, 1960). Particular subsystems of sequential quantum logic, which have their own importance, are represented by other well-known structures. The restriction of the system to logically connected propositions leads to an orthocomplemented quasimodular lattice structure [see Mittelstaedt (1978), p. 29]. Furthermore, a restriction of the system to particular logically connected propositions, which play a distinguished role in the framework of sequential quantum logic, leads to algebraic structures investigated by Kröger (1973, 1974) and Dishkant (1977). All these algebraic approaches to quantum mechanics can be distinguished systematically within the framework of the sequential quantum logic. Concerning a reconstruction of quantum physics, it is interesting to consider realizations of the extended quantum mechanical propositional system within the common Hilbert space formalism. Whereas the material and logically connected propositions are realized by the projection operators on a Hilbert space, it is shown that sequentially connected propositions are realized by products and particular sums of projection operators.

The extension of the proof theoretic semantics to sequential propositions is also desirable with regard to probability statements of the language of quantum mechanics. Most probability statements concern sequential probabilities which, in the framework of our approach, are probabilities of sequential propositions to be true. Probabilities of sequential propositions in the language of quantum physics have been considered in a brief report (Stachow, 1979b) and will be investigated in detail in a forthcoming article.

1. THE LANGUAGE OF QUANTUM PHYSICS

Before we consider a possible semantics of the language of quantum physics which appraises a logical system for quantum mechanical propositions to be adequate and understood, it should be made precise what we mean by a logical foundation of a physical theory.

1.1. The Logical Foundation of a Physical Theory

The subject of a logical foundation of a physical theory is the reconstruction of the theory by means of a *logical system*. We make use here of the standard terminology of logic texts [e.g., see van Fraassen (1971)], some basic notions of which may be defined in the following way.

(1.1) *Definition.* A *syntactic system* (Syn) is a pair $\langle V,E\rangle$, where
(a) V is a set, at most denumerable (the *vocabulary* or the set of *words*);
(b) E is a set (the set of *expressions* that are sequences of words and specified by a *grammar*; subsets of E are the sets of *nouns*, *sentences*, and *functors*).

An example is the syntactic system of the classical propositional logic:

(1.2) *Definition.* A *C-propositional syntactic system* (CPS) is a pair $\langle\langle S_a,C\rangle,S\rangle$, where
(a) S_a is the set of *atomic sentences*;
(b) C is the set $\{\wedge,\vee,\rightarrow,\neg,),(\}$ of *logical signs* [the *logical connectives* $\wedge$, $\vee$, $\rightarrow$, $\neg$, and *parentheses*), (];
(c) S is the set of *sentences*, recursively defined by (i) if $a\in S_a$ then $a\in S$; (ii) if $A,B\in S$ then $(A\wedge B),(A\vee B),(A\rightarrow B),(\neg A)\in S$.

Also the important notion of a *logical system* comprises a purely formal concept devoid of any interpretation of the system.

(1.3) *Definition.* A *logical system* (LS) is a pair $\langle \mathrm{Syn},\vdash\rangle$, where
(a) Syn is a syntactic system;
(b) $\vdash$ is a relation from sets of sentences of Syn to sentences of Syn ($X\vdash A \overset{\text{def.}}{\leftrightarrows} (X,A)\in\vdash$).

We define Th as the set $\{A\in S\colon \varnothing\vdash A\}$ (the set of *theorems*). The set of theorems of a logical system can be specified in a constructive manner by means of a *logical calculus*.

(1.4) *Definition.* A *logical calculus* (LC) is a triple $\langle \mathrm{Syn},\vdash,R\rangle$, where
(a) Syn is a syntactic system;
(b) $\vdash$ is a relation $\subseteq\mathcal{P}(S)\times S$, where $\mathcal{P}(S)$ is the power set of S ($X\vdash A$ is called a *figure*);
(c) R is a set consisting of the *beginnings* (or axioms) of the calculus, denoted by $\Rightarrow X\vdash A$, and also of the constitutive *rules* of the calculus, denoted by $X_1\vdash A_1,,\ldots,,X_n\vdash A_n\Rightarrow X\vdash A$.

Now we can define the set (F) of *deducible figures* of LC recursively by: (i) if $\Rightarrow X \vdash A \in R$ then $X \vdash A \in F$; (ii) if $X_1 \vdash A_1, \ldots, X_n \vdash A_n \in F$ and $X_1 \vdash A_1,, \ldots,, X_n \vdash A_n \Rightarrow X \vdash A \in R$, then $X \vdash A \in F$. The set of theorems of a logical system is specified by a logical calculus iff $\varnothing \vdash A \in \mathrm{Th} \leftrightarrows \varnothing \vdash A \in F$.

A logical system receives its appraisal only in connection with a *language* which distinguishes the logical system as an adequate presentation of the language.

(1.5) *Definition*. A *formal language* (L) is a pair $\langle \mathrm{Syn}, \mathrm{VL} \rangle$, where
(a) Syn is a syntactic system;
(b) VL is a nonempty set of functions v (the set of *admissible valuations* of Syn) which assign T (*true*) to some sentences of S, and/or F (*false*) to some sentences of S.

The specification of the admissible valuations of a formal language L usually is called the *formal semantics* of L. A well-known example is a bivalent semantics which establishes a *bivalent* propositional language.

(1.6) *Definition*. A *bivalent propositional language* is a pair $\langle \mathrm{CPS}, \mathrm{BVL} \rangle$, where
(a) CPS is the C-propositional syntactic system;
(b) BVL is the set of functions v (*bivaluations*), such that for all sentences $A, B \in \mathrm{CPS}$
(i) $v(A) \in \{\mathrm{T}, \mathrm{F}\}$;
(ii) $v(A \wedge B) = \mathrm{T}$ iff $v(A) = v(B) = \mathrm{T}$;
(iii) $v(\neg A) = \mathrm{T}$ iff $v(A) = \mathrm{F}$.

Important semantic concepts are those of *satisfaction* and *validity*. A sentence A of L is said to be *satisfied* by an admissible valuation v if and only if $v(A) = \mathrm{T}$. A is said to be *satisfiable* if and only if there exists an admissible valuation v such that $v(A) = \mathrm{T}$. A sentence A of L is said to be *valid* ($\Vdash A$) if and only if $v(A) = \mathrm{T}$ for all admissible valuations v of L. The adequateness of a logical system with respect to a language can be made precise now by the correspondence: $\Vdash A$ in L $\leftrightarrows$ $\varnothing \vdash A$ in LS ("$\rightharpoondown$" yields the *completeness*, "$\leftharpoondown$" yields the *soundness* of LS with respect to L).

These concepts of logic and formal semantics are important for the consideration of the language of physics. They are used here to establish an adequate logical system for the language of quantum mechanics.

A physical theory, like quantum mechanics, uses mathematical structures which are more concrete and more common than the abstract structure of the logical system of a physical language. However, it is the purpose of this paper to show that there is a relation between these structures in the sense that the structure of the logical system is realized by (at least a part of) the mathematical formalism of the theory. In this way, which we call a logical foundation of a physical theory, one can perhaps better understand why a particular physical theory is so effective in describing actual operations.

1.2. The Syntax of the Language

The formal language of quantum physics comprises a syntactic system and a set of admissible valuations for its sentences. In order to establish this formal language as an adequate language for quantum physics, the elements of the syntax should be determined to be symbols for *meaningful* physical and linguistic concepts. We call such an interpretation of the syntax of L also an *underlying semantics*. The underlying semantics then leads to a specification of the admissible valuations of the language. In our approach, the syntax of the language of quantum physics is specifically motivated by and adapted to the *pragmatic operations* of a *science*. This approach to a formal language has been carried out and will be extended here to a richer syntax in a systematic way.

1.2.1. Elementary Propositions. Elementary propositions belong to the set of atomic sentences of the syntax of the language. They are predictions of particular properties of a quantum mechanical system. We assume that elementary propositions can be proved or disproved by means of operational tests. Let us consider the following example. An electron ($\mathbb{S}$) is given which moves in the $+z$ direction of a coordinate system. An elementary proposition is "The electron has 'spin-up' in the direction a in the x–y plane at the time t," in symbols "$a(\mathbb{S}, t)$," where t is the future point of time with respect to which the property is predicted. An operational test of this proposition could be a Stern–Gerlach experiment, which is oriented in the direction a, together with a photographic plate. The mark, which the electron leaves on the plate, indicates the deviation of the electron in the magnetic field and, thus, proves or disproves the proposition under consideration. Of course, this description of an operational test is a rather simplified and imprecise description. A careful and detailed analysis of a physical operation, provided with a mathematical model, is given by Randall and Foulis (1976, 1979), Ludwig (1977). However, we accept the view that a precise account of what a physical operation means can be given only in the context of an established physical theory, or in the context of an already established language of physics. Hence we are led to the following commitment. The operational proof procedures for elementary propositions, which belong to the underlying semantics of the language to be established, must be described precisely and completely by the language in the final stage when the language is established. This demand with respect to an operational approach to a physical theory is called *semantical consistency* (v. Weizsäcker) or *self-consistency* (Mittelstaedt). We cannot here investigate the difficult problem of the consistency of the concept of elementary propositions with respect to an underlying semantics. Since we are, in the first place, interested in a formal logical system which is adequate to the language, we *assume* that elementary propositions are described by the Hilbert space formalism consistently as it is explained in Section 1.3.1. We call this specific example of an underlying semantics for elementary propositions a *Hilbert space semantics.*

Elementary propositions are denoted by $a, b, \ldots$; the at most denumerable set of elementary propositions is denoted by S_e. The underlying semantics precisely formulates the proof conditions for elementary propositions. In this case we say that elementary propositions are *proof definite.* If, in addition, it is defined when an elementary proposition a is disproved, we say that a is *disproof definite.* If, in the sense of semantical consistency, there exists a proof procedure for a, such that, after performing this procedure, a is either proved or disproved, we say that a is *value definite.* If, furthermore, this proof procedure, when repeated immediately, always leads to the same result, we say that a is *idempotent.* In the following we assume that all elementary propositions are idempotent. This assumption is consistent with the Hilbert space semantics considered in Section 1.3.1.

It is useful to introduce two additional propositions V (the *true proposition*) and Λ (the *false proposition*) which are defined by the conditions that V is *valid* and Λ is *not satisfiable* (see Section 1.1).

The set of atomic sentences of the language of quantum physics is then given by $S_a = S_e \cup \{V, \Lambda\}$.

1.2.2. Compound Propositions. An essential concept that is considered in this paper is that of a compound proposition. Since we are in the possession of a proof theory for elementary propositions, the question arises how sentences that are composed of elementary propositions can meaningfully be defined by an underlying semantics. Usually, a propositional syntax includes the *conjunction* $\wedge$, the *disjunction* $\vee$, the *material implication* $\rightarrow$, and the *negation* $\neg$ (which are in a certain sense understood as the colloquial *and*, *or*, *if–then*, and *not*, respectively) as logical connectives. The interpretations of these logical connectives in a language of quantum physics has led to many anomalies with respect to the concepts of classical propositional logic and semantics. In the framework of an underlying *operational* semantics, logically connected propositions should permit an operational proof procedure which recurs to specific proofs and disproofs of the elementary subpropositions. Intuitively, one would stipulate for the proof condition of a sentence $a \wedge b$ that the elementary proposition

a and the elementary proposition b must be proved, where the "and" in this sentence is meant in a pragmatic sense. As an example, let us consider two elementary propositions of the sort $a(\mathbb{S},t)$, $b(\mathbb{S},t)$ as defined above with $a \neq b$. In any operational procedure that includes tests of a and b, these tests are performed in a sequence. According to our example, where elementary propositions are associated (in an idealizing way) with points of time, the tests of a and b refer to two different time values of a and b, for instance $a(\mathbb{S},t_1)$ and $b(\mathbb{S},t_2)$ with $t_1 < t_2$. Now, the pair (a,b) has a particular property which we call *incommensurability*. This property means that there exists no measurement for a which, after a sequence of measurements of a and then b, reproduces the result with certainty; analogously for b. For a proof of the sentence $a \wedge b$, a sequence of measurements of a and b should not be sufficient. Only an additional proof of the *commensurability* of the pair (a,b) should establish a proof of $a \wedge b$. In our example, only the commensurability would justify referring a and b to the same point of time and, thus, using a sequence of measurements of a and b in order to prove the sentence $a(\mathbb{S},t) \wedge b(\mathbb{S},t)$. However, because of the incommensurability of a and b, the proposition $a(\mathbb{S},t) \wedge b(\mathbb{S},t)$ can never be proved, even if $a(\mathbb{S},t)$ and $b(\mathbb{S},t+\Delta t)$ can be proved in an immediate succession with $\Delta t \to 0$ [see Mittelstaedt (1978), Chap. 3]. In this way, it can easily be seen that an underlying operational semantics for logically connected propositions does not specify a bivalent propositional language (1.6) of quantum physics. As a consequence, various attempts have been made to establish many-valued logics for quantum mechanics. We do not consider them here, but rather offer another approach which uses the two values *true* (T) and *false* (F) only.

Our approach consists in a systematic construction of the language of quantum physics as an argumentation procedure with respect to stated sentences. This means that (for our purposes) we restrict the set S of sentences to those sentences which, in the actual language, are interpreted as statements that are open to doubts and, upon a doubt, must be justified in some way. Therefore we use the term "proposition" instead of "sentence." The process of argumentation is formulated as a language game. Particular possibilities of argumentation, represented by means of the argument rules of a dialog game, define the connective structure of compound propositions. Since the dialog game provides the compound propositions with a proof procedure, compound propositions are said to be *dialog definite*. There are, in addition to the logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, three kinds of 2-place *sequential connectives*, the *sequential conjunction* $\sqcap$, the *sequential disjunction* $\sqcup$, and the *sequential material implication* $\dashv$ (in a certain sense specifying the colloquial *and then*, *or then*, and *if first–then*, respectively). Each sequential connective represents particular possibilities of the succession of two arguments within the language game. An example of a sequentially connected proposition $a \sqcap b$ is the proposition $a(\mathfrak{S}, t) \sqcap b(\mathfrak{S}, t+\Delta t)$ with $\Delta t > 0$, which comprises a sequence of measurements of a and b at the time t and at the time $t+\Delta t$, respectively, as was discussed above. In order to prove logically connected propositions within a finite game, we need *commensurability propositions* $k(a,b)$ which state the commensurability of the pair of propositions (a,b). This is made precise by the following definition.

(1.7) *Definition.* The *commensurability proposition* $k(a,b)$ states that for all sequences of proofs of a and b it is satisfied that the results of the proofs of a are all the same, and the results of the proofs of b are all the same.

The commensurability and incommensurability, respectively, of a pair (a,b) are considered here to be properties of a quantum mechanical system which can be proved by experimental tests. Since they are propositions about measurable properties of a system, the commensurability propositions have the same status as elementary propositions. Therefore both kinds of propositions are called *material propositions*. We assume that also for commensurability propositions the Hilbert space formalism of quantum mechanics provides an underlying semantics which formulates the proof conditions of these propositions in a precise way, as is worked out in Section 1.3.1.

1.2.3. The Syntax of the Formal Language. The above considerations suggest the following syntax of the formal language of quantum physics.

(1.8) *Definition.* A *Q-propositional syntactic system* (QPS) is a pair $\langle\langle S_e, C\rangle, S\rangle$, where

(a) S_a is the set consisting of the *elementary propositions*, denoted by $a, b, \ldots$, the *true proposition* V and the *false proposition* Λ;

(b) C is the set $\{\wedge, \vee, \rightarrow, \neg, \sqcap, \sqcup, \dashv, k(\ ,\ ),), (\}$ of the *logical* and *sequential connectives*, the *commensurability sign* $k(,)$, and the *parentheses*), (;

(c) S is the set of *sequential propositions* (*S propositions*), denoted by $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, which includes the set L of *logical propositions* (*L propositions*), denoted by $A, B, \ldots$. S is recursively defined by
   (i) if $a \in S_e$ then $a \in L$;
   (ii) if $A, B \in L$ then $(A \wedge B), (A \vee B), (A \rightarrow B), (\neg A), k(A,B) \in L$;
   (iii) if $A \in L$ then $A \in S$;
   (iv) if $\mathfrak{A}, \mathfrak{B} \in S$ then $(\mathfrak{A} \sqcap \mathfrak{B}), (\mathfrak{A} \sqcup \mathfrak{B}), (\mathfrak{A} \dashv \mathfrak{B}), \neg \mathfrak{A} \in S$.

The above syntactic system contains a comprehensive set of connectives. Not all connectives are independent if the *value definiteness* of material propositions is assumed. As is the case for the C-propositional syntactic system (1.2), an arbitrary pair of the 2-place logical connectives $\wedge, \vee, \rightarrow$ can be eliminated. Moreover, in QPS, all 2-place logical connectives and an arbitrary pair of sequential connectives can be eliminated. The connectives $\sqcap, \neg$ would be a minimal basis of connectives, for instance.

1.3. Proof Theoretic Semantics of the Language

We emphasize that the term "proof theoretic," as it is used here, characterizes a particular underlying semantics of the language which interprets the sentences by means of their proof possibilities. These proof possibilities are formulated by a theory (for instance the Hilbert space theory for material propositions, the theory of dialog games for compound propositions). It must be distinguished from the "proof-theoretic" methods of the metalogic which deal with the specific setup of axioms and rules for a formulation of a logical system.

1.3.1. Hilbert Space Semantics of Material Propositions. As an example of a semantics of material propositions we refer to the Hilbert space formalism of quantum mechanics. A physical system $\mathbb{S}$ is represented by a Hilbert space $\mathcal{H}(\mathbb{S})$ (in general by a family of Hilbert spaces $\mathcal{H}_\alpha(\mathbb{S})$, but we restrict our consideration here to the simple case of one Hilbert space). Each property $\hat{a}$ of the system is associated with a closed subspace M_a of $\mathcal{H}(\mathbb{S})$. Depending on the particular preparation, the system is characterized by a particular state. A system can always be prepared in a pure state which corresponds to a vector $|\varphi\rangle$ in the Hilbert space.

If a system is given in a state $|\varphi\rangle$, a measurement of a property $\hat{a}$ is represented by a mapping which projects the vector $|\varphi\rangle$ orthogonally into the subspace M_a. This mapping defines the projection operator P_a on $\mathcal{H}$ with the range M_a. Hence, each property $\hat{a}$ of the system can equally be associated with the projection operator P_a.

In case P_a satisfies the relation $P_a|\varphi\rangle = |\varphi\rangle$, which is equivalent to $|\varphi\rangle \in M_a$, the property $\hat{a}$ will certainly be proved by a measurement. In case $P_a|\varphi\rangle = 0$ holds, which is equivalent to $|\varphi\rangle \in M_a^\perp$ where $M_a^\perp$ is the completely orthogonal subspace with respect to M_a, the property $\hat{a}$ will certainly by disproved by a measurement. In these two cases, the Hilbert space formalism permits inferring the truth and falsity of an elementary proposition (which is a prediction of a property of the system) already from the preparation of the system.

However this is not possible for all elementary propositions. If $|\varphi\rangle \notin M_a$ and $|\varphi\rangle \notin M_a^{\perp}$, the theory of measurement does not specify the system to be in a pure state after the interaction with a measuring apparatus, but in a mixed state [e.g., see Mittelstaedt (1976) Chap. III]. This mixed state comprises the two possibilities that, after "reading the scale," the system is in the state $|\psi\rangle$, which is the orthogonal projection of $|\varphi\rangle$ into M_a, or in the state $|\chi\rangle$, which is the orthogonal projection of $|\varphi\rangle$ into $M_a^{\perp}$.

The Hilbert space formalism leads to the following proof and disproof conditions for elementary propositions. Let us assume that a system $\mathbb{S}$ is prepared in the state $|\varphi\rangle$, denoted by $\mathbb{S}(\varphi)$. An elementary proposition states a certain property $\hat{a}$ of the system after its preparation, denoted by $a(\varphi)$. If, after the measuring process with respect to the property $\hat{a}$ (which includes reading the scale), the system is in the state $|\varphi\rangle$, we have the following.

(1.9) *Definition.*

(a) $v(a(\varphi)) = \mathrm{T} \leftrightarrow a(\varphi)$ is *proved* $\overset{\text{def.}}{\leftrightarrow} P_a|\varphi'\rangle = |\varphi'\rangle$;

(b) $v(a(\varphi)) = \mathrm{F} \leftrightarrow a(\varphi)$ is *disproved* $\overset{\text{def.}}{\leftrightarrow} P_a|\varphi'\rangle = 0$.

The Hilbert space semantics is extended to commensurability propositions. Each commensurability proposition $k(a,b)$ is associated with the closed subspace $(M_a \cap M_b) \cup (M_a \cap M_b^{\perp}) \cup (M_a^{\perp} \cap M_b) \cup (M_a^{\perp} \cap M_b^{\perp}) = M_{k(a,b)}$, where the operation $\cap$ denotes the intersection of subspaces and the operation $\cup$ denotes the span of subspaces. The correspondence of $k(a,b)$ to $M_{k(a,b)}$ may be motivated at this stage by the relation

$$(1.10) \qquad |\varphi\rangle \in M_{k(a,b)} \leftrightarrow [P_a, P_b]_-|\varphi\rangle = 0$$

which can easily be proved in the Hilbert space formalism. The projection operator $P_{k(a,b)}$ onto $M_{k(a,b)}$ is given[1] by

$$P_{k(a,b)} := \underset{n\to\infty}{\text{(s)lim}}\,(P_a \cdot P_b)^n + \underset{n\to\infty}{\text{(s)lim}}\,(P_a \cdot (1-P_b))^n + \underset{n\to\infty}{\text{(s)lim}}\,((1-P_a)\cdot P_b)^n$$
$$+ \underset{n\to\infty}{\text{(s)lim}}\,((1-P_a)\cdot(1-P_b))^n$$

such that

$$(1.11) \qquad P_{k(a,b)}|\varphi\rangle = |\varphi\rangle \leftrightarrow |\varphi\rangle \in M_{k(a,b)}$$

$$(1.12) \qquad P_{k(a,b)}|\varphi\rangle = 0 \leftrightarrow |\varphi\rangle \in M_{k(a,b)}^{\perp}$$

[1]The symbol (s)lim denotes the *strong convergence* of a sequence of operators.

Since we assume that $k(a,b)$ can be proven by an experimental test, the above considerations of the theory of measurement can immediately be applied to commensurability propositions also:

(1.13) *Definition.*

(a) $v(k(a,b)(\varphi)) = \mathrm{T} \leftrightarrows k(a,b)(\varphi)$ is *proved* $\overset{\text{def.}}{\leftrightarrows} P_{k(a,b)}|\varphi'\rangle = |\varphi'\rangle$;

(b) $v(k(a,b)(\varphi)) = \mathrm{F} \leftrightarrows k(a,b)(\varphi)$ is *disproved* $\overset{\text{def.}}{\leftrightarrows} P_{k(a,b)}|\varphi'\rangle = 0$.

The various connections between material propositions and closed subspaces of a Hilbert space, operational proofs of material propositions by means of measurements and the theory of quantum mechanical measurements are supposed to satisfy the principle of *semantical consistency*, and are summarized in Diagram 1.

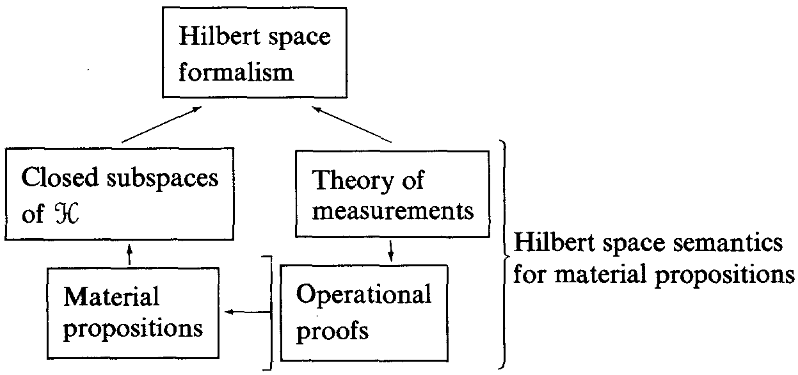


Diagram 1

1.3.2. Dialogic Semantics for Compound Propositions. The concepts of logical and sequential connectives of the language of quantum physics are meaningfully defined by the possibilities of argumentation within quantum mechanical language games. Since the theory of dialog games also determines the proof conditions for logically and sequentially connected propositions, it provides the compound quantum mechanical propositions with an underlying semantics.

It should be noted that the language game for quantum mechanical propositions has a structure that can be founded without reference to the particular empirical content of quantum mechanical propositions (Stachow, 1976, 1979a). In these papers it is argued that a language for science has a certain discoverable skeleton that can adequately be reconstructed by means of a language game for science. This skeleton, as a *universal structure* of a scientific language, is a linguistic precondition of any scientific conception of reality. In this sense it may be considered as a cognition that is a priori valid. It is a remarkable feature of the formal language of quantum physics that it is entirely established by this language game.

In the following, the language game for quantum mechanical propositions is presented in a systematic manner without considering questions of its justification (which can be found in the above-cited articles). The language game can be formulated as a *dialog game* [or a *two-person zero-sum game* in the terminology of game theory, e.g., Berge (1957)]. The rules of the dialog game are divided into two classes: the *frame rules*, which constitute the concept of a dialog, and the *argument rules*, which specify the possibilities of argumentation within the framework of a dialog.

The *frame rules* of the dialog game are as follows:

F1: At the beginning of the dialog, the proponent (P) asserts the initial argument. In this way, the initial position of the dialog game is established.

F2: After the assertion of the initial argument, an opponent (O) may attack this argument. Thereupon, the proponent is obliged to defend the initial argument against the attack.

F3: The dialog consists in a sequence of arguments which are assertions, attacks, and defences of the two participants.

F4: If one of the two participants cannot continue to put forth an argument, he loses the dialog. In this case the other one wins, and the final position of the dialog is established.

(1.14) *Definition. Arguments* that may be used in a dialog are

(a) *S propositions* (1.8); the initial argument of a dialog always is a proposition;

(b) the *challenges* $1?, 2?, ?$ to state a subproposition of a compound proposition; they are always used as attacks;

(c) the *challenges* $a!, k(A,B)?$ to prove elementary and commensurability propositions, respectively; they are always used as attacks;

(d) the *proof arguments* $a?, k(A,B)!$, symbolizing a proof of a and of $k(A,B)$, respectively; they are always used as defences.

The *argument rules* of the dialog game are

A′1: (a) If, in a dialog about a compound proposition, a subproposition is asserted, the assertion is taken to be an initial argument in a new dialog, a subdialog, about the subproposition.

(b) The possibilities of argumentation with respect to logically connected propositions are

(i) If $A \wedge B$ is asserted, it may be attacked by a challenge $\in \{1?, 2?\}$. Upon the attack 1?, the obligation to defend consists in the winning of a subdialog about A; upon the attack 2?, the obligation to defend consists in the winning of a subdialog about B. Upon a defence, the proposition $A \wedge B$ may be attacked again.

(ii) If $A \vee B$ is asserted, it may be attacked by the challenge ?. Upon the attack ?, the obligation to defend consists in the winning of a subdialog with respect to one of the subpropositions $\in \{A, B\}$. If an attempt to defend fails by the loss of the subdialog, the participant may try again to defend by winning a subdialog.

(iii) If $A \rightarrow B$ is asserted, it may be attacked by the assertion of the subproposition A. Upon the attack, the obligation to defend consists in the assertion of the subproposition B, but is postponed according to (a) until the subdialog about A is won by the participant who attacked by A. In case the subdialog about A is lost, there is no longer an obligation to defend. Upon a defence, the proposition $A \rightarrow B$ may be attacked again.

(iv) If $\neg A$ is asserted, it may be attacked by the assertion of the subproposition A. Upon the attack, no defence is possible.

The possibilities of attack and defence are summarized in Table I.

(c) The possibilities of argumentation with respect to sequentially connected propositions are given by Table II, which is self-explanatory. Whereas with respect to logically connected propositions, the dialog is ruled by *possible* successions of attacks and of defences, the successions of

TABLE I

Logical connective	Logically connected proposition	Possibilities of attack	Possibilities of defence
conjunction	$A \wedge B$	1? 2?	A B
disjunction	$A \vee B$	?	A B
material implication	$A \rightarrow B$	A	B
negation	$\neg A$	A	

TABLE II

Sequential connective	Sequentially connected proposition	Attacks	Defences
sequential conjunction	$\mathfrak{A} \sqcap \mathfrak{B}$	1. 1? 3. 2?	2. $\mathfrak{A}$ 4. $\mathfrak{B}$
sequential disjunction	$\mathfrak{A} \sqcup \mathfrak{B}$	1. ?	2. $\mathfrak{A}$ 3. $\mathfrak{B}$
sequential material implication	$\mathfrak{A} \dashv \mathfrak{B}$	1. $\mathfrak{A}$	2. $\mathfrak{B}$
negation	$\neg \mathfrak{A}$	1. $\mathfrak{A}$	

attacks and of defences are uniquely determined in dialogs with respect to sequentially connected propositions. The succession of the arguments is indicated by numbers in the table.

A_m2: If an elementary proposition a is asserted in a dialog, it may be attacked by the argument a?. The obligation to defend consists in a proof of a which is performed outside of the dialog by means of a measuring process. If a proof of a is established, this is indicated by the defence argument a! in the dialog.

The truth and falsity of a compound proposition should not depend on the win and loss, respectively, of the individual run of a dialog. It might be the case that a particular choice among the possibilities of argumentation with respect to a logically connected proposition leads to winning the dialog, whereas another choice would not. In order to establish the truth and falsity of a logically connected proposition, the contingency of the win and loss of a dialog due to the particular choice of the arguments should be excluded. This can be done by means of commensurability propositions as is indicated below. It is useful to establish a dialog game in which the win by the proponent establishes the truth, and the loss by the proponent establishes the falsity of the initial argument. Such a dialog game is called the *material dialog game*. It is obtained by replacing the argument rule A′1 by the new argument rule A1 which differs from A′1 in (b) only, and by adding the additional argument rule A_m3:

A1: (b) A participant wins the dialog about a logically connected proposition, irrespective of the particular choices among the possibilities of argumentation (we say he has a *strategy of success*), if and only if he wins the dialog which makes use of the attacks and defences given by Table III.

A_m3: If a commensurability proposition $k(A,B)$ is asserted in a dialog, it may be attacked by the argument $k(A,B)$?. The obligation to defend consists in a proof of $k(A,B)$ which is performed outside of the

TABLE III

Logical connective	Logically connected proposition	Attacks	Defences
conjunction	$A \wedge B$	1. 1? 3. 2! 5. $k(A,B)$?	2. A 4. B 6. $k(A,B)$
disjunction	$A \vee B$	1. !	2. A 3. B 4. $\neg k(A,B)$
material implication	$A \rightarrow B$	1. A 3. $k(A,B)$?	2. B 4. $k(A,B)$
negation	$\neg A$	1. A	

dialog by means of a measuring process. If a proof of $k(A,B)$ is established, this is indicated by the defence argument $k(A,B)!$ in the dialog.

The *material dialog game* (D_m) is defined by the rules F1–F4 and A1–A_m3. A detailed discussion of this dialog game can be found in the book by Mittelstaedt (1978), Chap. 3. If the proponent wins the material dialog about the proposition $\mathcal{A}$, this is denoted by $\vdash_{D_m} \mathcal{A}$.

The proof and disproof conditions for compound propositions can now be specified by the following definition.

(1.15) *Definition.*

(a) $\mathcal{A}$ is *materially true* $\Leftrightarrow \mathcal{A}$ is *proved* $\overset{\text{def.}}{\Leftrightarrow} \vdash_{D_m} \mathcal{A}$;

(b) $\mathcal{A}$ is *materially false* $\Leftrightarrow \mathcal{A}$ is *disproved* $\overset{\text{def.}}{\Leftrightarrow} \vdash_{D_m} \neg \mathcal{A}$.

It is clear from the material dialog game that a proof or disproof of a compound proposition is partly a result of the rules or argumentation and partly a result of measuring processes with respect to a quantum mechanical system under consideration. Under the *dialog semantics* we understand the pure structure of argumentation as it is stipulated by the structural rules F1–F4 and A1 of the material dialog game. The dialogic semantics, together with the Hilbert space semantics, establishes a proof theory for compound quantum mechanical propositions in a way that is illustrated by means of Diagram 2.

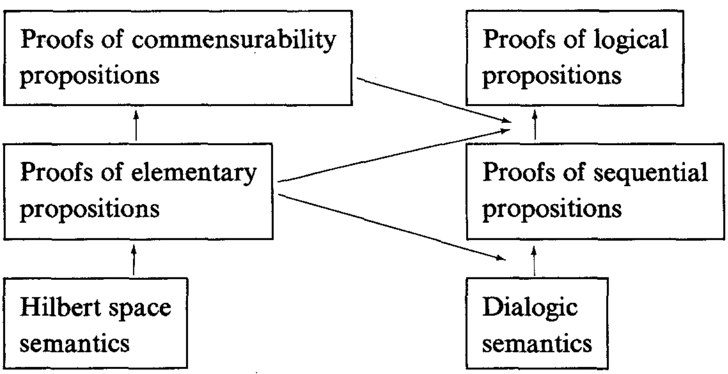
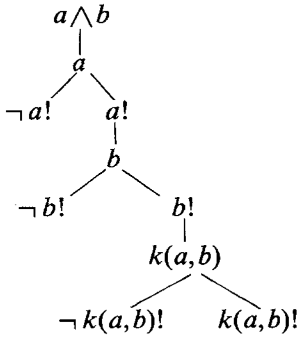


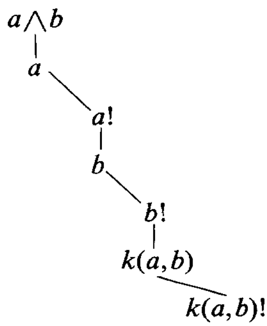
Diagram 2

1.3.3. Admissible Valuations of the Language. The proof-theoretic semantics is now used for specifying the set of admissible valuations of the language of quantum physics. As we shall show in the following, the dialogic proof procedure provides quantum mechanical propositions with a very natural valuation.

Each dialog consists in a succession of arguments which is determined by the dialog rules up to the tests of material propositions. Because of the contingent proof results of material propositions, there are different possibilities of the continuation of the dialog. These possibilities may be represented by a game tree. As an example let us consider the game tree with respect to the proposition $a \wedge b$:



Disproofs of material propositions m are here denoted by $\neg m!$. Each individual dialog about $a \wedge b$ runs from the initial argument $a \wedge b$ to a final argument of the tree, depending on the contingent sequence of proof results of the material propositions. A possible dialog about $a \wedge b$ is for instance



If the system is initially prepared in the state $|\varphi\rangle$, and the state after the measuring process with respect to a is $|\varphi'\rangle$, the state after the measuring process with respect to b is $|\varphi''\rangle$ and the state after the measuring process with respect to $k(a,b)$ is $|\varphi'''\rangle$, we have for the above dialog

$$P_a|\varphi'\rangle = |\varphi'\rangle, \qquad P_b|\varphi''\rangle = |\varphi''\rangle, \qquad P_{k(a,b)}|\varphi'''\rangle = |\varphi'''\rangle$$

In this case, the proposition $a \wedge b$ is true. In the remaining three cases of possible dialogs about $a \wedge b$, the proposition is false.

Since the transition of a physical system, due to a material dialog, is characterized by the sequence of proofs and disproofs of the material propositions, we call such a sequence a *possible material process*. Each dialog is specified by a possible material process which is the contingent dynamical part of the dialog, and a structure of argumentation which is the linguistic and static part of the dialog. Let us assume now that, with respect to a physical system $\mathbb{S}$, there exists the set of all possible material dialogs and, thus, the set ($P(\mathbb{S})$) of all possible material processes. Each proposition $\mathfrak{A}$ about a system is associated with a set ($P(\mathfrak{A})$) of possible material processes which is a subset of $P(\mathbb{S})$. By means of this semantical postulate with respect to the dialog game we arrive at the following definition.

(1.16) *Definition*. (a) A proposition $\mathfrak{A}$ is *true in* $p \in P(\mathfrak{A}) \overset{\text{def.}}{\leftrightsquigarrow}$ the proponent wins the dialog about $\mathfrak{A}$ under the condition that the sequence of proofs and disproofs of material propositions is p.

(b) A proposition $\mathfrak{A}$ is *false in* $p \in P(\mathfrak{A}) \overset{\text{def.}}{\leftrightsquigarrow}$ the proponent loses the dialog about $\mathfrak{A}$ under the condition that the sequence of proofs and disproofs of material propositions is p.

A valuation of the propositions of the formal language can now be obtained by the following.

(1.17) *Definition*. A *valuation over* $P(\mathbb{S})$ is the function

$$v\colon M \subseteq P(\mathbb{S}) \times S \rightarrow \{\mathrm{T}, \mathrm{F}\}$$

with

$$M := \{(p, \mathfrak{A}) : \mathfrak{A} \in S, p \in P(\mathfrak{A})\}$$

and

(a) $$v_p(\mathfrak{A}) := v((p, \mathfrak{A})) = \mathrm{T} \overset{\text{def.}}{\leftrightsquigarrow} \mathfrak{A} \text{ is true in } p;$$

(b) $$v_p(\mathfrak{A}) = \mathrm{F} \overset{\text{def.}}{\leftrightsquigarrow} \mathfrak{A} \text{ is false in } p.$$

The function

$$v_p\colon \Delta_p \subseteq S \rightarrow \{\mathrm{T}, \mathrm{F}\}$$

with

$$\Delta_p := \{\mathfrak{A} \in S : p \in P(\mathfrak{A})\}$$

is called an *admissible valuation on* S. The set of admissible valuations on S (QVL) is the set $\{v_p\colon p \in P(\mathbb{S})\}$.

This establishes the *formal language of quantum physics* (QL) as a pair ⟨QPS, QVL⟩ where QPS is the *Q-propositional syntactic system* (1.8) and QVL is the set of admissible valuations of S. The concept of *satisfaction* can also be defined by means of our proof theoretic semantics. A proposition $\mathfrak{a}$ is said to be *satisfied* by v_p if and only if $v_p(\mathfrak{a})=\mathrm{T}$. $\mathfrak{a}$ is said to be *satisfiable* if and only if there exists a $p \in P(\mathbb{S})$ such that $v_p(\mathfrak{a})=\mathrm{T}$. (The concepts of *refutation* and *refutability* can be defined analogously.) However, the concept of *validity* cannot be defined as usual (see 1.1), since a proposition $\mathfrak{a}$ does not belong to the domain of all admissible valuations. Therefore, we say that a proposition $\mathfrak{a}$ is *always true* (or *not refutable*) if and only if $v_p(\mathfrak{a})=\mathrm{T}$ for all $p \in P(\mathfrak{a})$ (*always false* or *not satisfiable* analogously).

Our considerations with respect to the language of quantum physics may be summarized by means of Diagram 3.

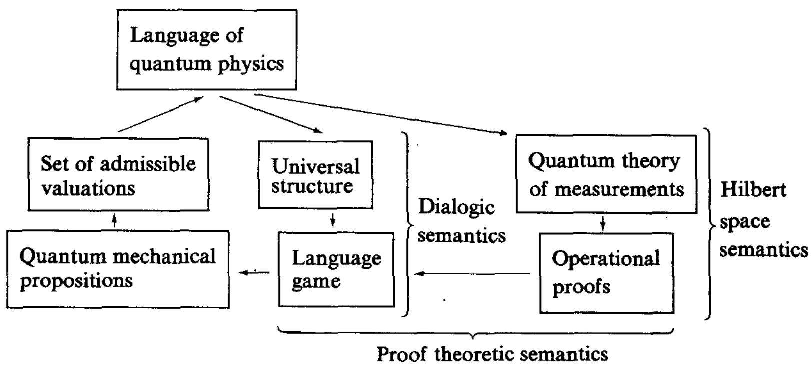


Diagram 3

2. THE SYSTEM OF QUANTUM LOGIC

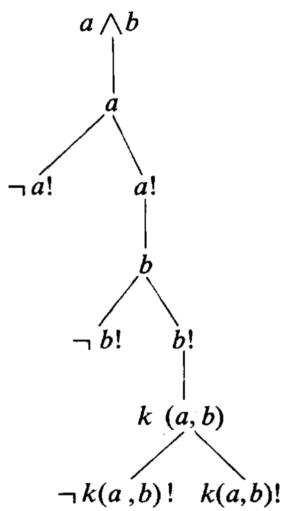
We are now concerned with a foundation of quantum logic which is based on the established language of quantum physics. A logical system, the system of *formal quantum logic*, is presented, which is adequate to the language in the sense that a proposition $\mathfrak{a} \in S$ is *formally true* [see Def. (2.8) in Section 2.2] iff $\mathfrak{a}$ is a theorem of the logical system. The proofs of the completeness and consistency of the logical system with respect to the language are given somewhere else because of their length and technical complexity.

Before considering the formal logic, we introduce an algorithm which replaces the material dialog game as a proof procedure for propositions that are true in a given material process.

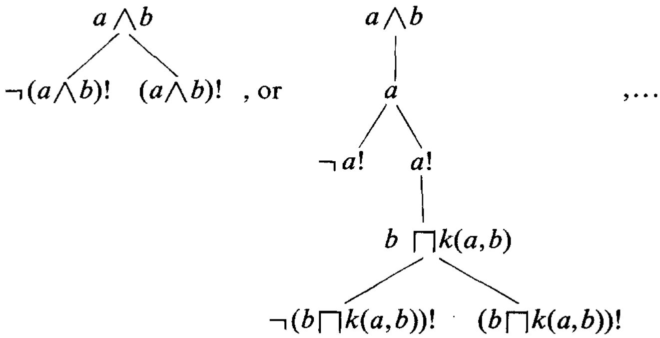
2.1. The Material Logic

A possible material process $p \in P(\mathbb{S})$ consists in a particular sequence of proofs and disproofs of elementary and commensurability propositions. According to the definition of the sequential conjunction (A1), p establishes the proof of an iterated sequential conjunction, denoted by $\sqcap p$, of the material propositions (proved) and the negations of the material propositions (disproved) in p. Hence, in our above example of a material dialog about $a \wedge b$, which is won by the proponent, the corresponding material process $\langle a!, b!, k(a,b)! \rangle$ immediately proves the proposition $(a \sqcap b) \sqcap k(a,b)$. On the other hand, the remaining three possible material processes $\langle \neg a! \rangle$, $\langle a!, \neg b! \rangle$, $\langle a!, b!, \neg k(a,b)! \rangle$ with respect to $a \wedge b$ immediately prove the propositions $\neg a$, $a \sqcap \neg b$, $(a \sqcap b) \sqcap \neg k(a,b)$, respectively. In this way, any possible material process p can uniquely be associated with a sequential conjunction $\sqcap p$.

As the auxiliary notion we define a *possible dialogic process* to be a sequence of proofs and disproofs of S propositions, denoted by d, which can be established in a dialog. The proofs of S propositions $\mathfrak{A}$ within the sequence are denoted by $\mathfrak{A}!$, the disproofs by $\neg \mathfrak{A}!$. The set of all possible dialogic processes with respect to a physical system $\mathbb{S}$ is designated by $D(\mathbb{S})$. We have $P(\mathbb{S}) \subseteq D(\mathbb{S})$. Dialogic processes are used as short forms for several possible material processes within a game tree. They represent a system of branchings by simpler systems, e.g., the game tree for $a \wedge b$:



can be represented by

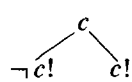


A set of possible dialogic processes which establishes a short form of a game tree for the proposition $\mathscr{A}$ is denoted by $D(\mathscr{A})$, e.g., $\{\langle(a \wedge b)!\rangle, \langle\neg(a \wedge b)!\rangle\}$. The set of all $D(\mathscr{A})$ which represent the same game tree can be constructed.

For the following, two relations between $P(\mathbb{S})$ and $D(\mathbb{S})$ are of importance. We assume that there exists a relation $\equiv_m \subseteq P(\mathbb{S}) \times D(\mathbb{S})$, called *material identity*, such that for $p \in P(\mathbb{S})$ and $d \in D(\mathbb{S})$

(2.1) $p \equiv_m d$, if and only if the proposition $\sqcap d$ may be replaced by $\sqcap p$ in any dialog about a proposition $\sqcap d'$ ($d' \in D(\mathbb{S})$) in which $\sqcap d$ occurs as a subproposition, without thereby influencing the possibility of winning the dialog about $\sqcap d'$.

If there exists an elementary proposition c such that $\langle c!\rangle \equiv_m \langle(a \wedge b)!\rangle$, the above system of branchings may be replaced by the one branching:



in *any* game tree.

In addition we assume that there exists a relation $=_m \subseteq P(\mathbb{S}) \times D(\mathbb{S})$, called *material equality* such that for $p \in P(\mathbb{S})$ and $d \in D(\mathbb{S})$:

(2.2) $p =_m d$, if and only if the proposition $\sqcap d$ may be replaced by $\sqcap p$ in any dialog in which $\sqcap d$ is asserted, such that the possibilities of winning the subdialogs about $\sqcap d$ and $\sqcap p$ are the same.

If there exists an elementary proposition c such that $\langle c!\rangle =_m \langle (a \wedge b)!\rangle$, the above sequence of branchings may be replaced by the one branching only *at the end* of a game tree.

Obviously we have $\equiv_m \subseteq =_m$.

The class $\{p' \in P(\mathfrak{S}) : p' \equiv_m p\}$ is denoted by $[p]_m$ and the set of all such classes $\{[p]_{\equiv} : p \in P(\mathfrak{S})\}$ is denoted by $P_{\equiv}(\mathfrak{S})$. In case a set $D(\mathfrak{A})$ of possible dialogical processes which represent a game tree for $\mathfrak{A}$ satisfies that for all $d \in D(\mathfrak{A})$ there exists a $p \in P(\mathfrak{S})$ such that $p \equiv_m d$, we define the set $P_{\equiv}(\mathfrak{A}) := \{[p]_{\equiv} \in P_{\equiv}(\mathfrak{S}) : p \equiv_m d,\ d \in D(\mathfrak{A})\}$.

An admissible valuation on the set S of propositions can now also be specified by the function $v_{[p]_{\equiv}} : \Delta_{[p]_{\equiv}} \subseteq S \to \{\mathrm{T}, \mathrm{F}\}$ with $\Delta_{[p]_{\equiv}} := \{\mathfrak{A} \in S$: there exists a $P_{\equiv}(\mathfrak{A})$ with $[p]_{\equiv} \in P_{\equiv}(\mathfrak{A})\}$, and the set of admissible valuations can be specified by $\{v_{[p]_{\equiv}} : [p]_{\equiv} \in P_{\equiv}(\mathfrak{S})\}$.

Analogously, these definitions can be transferred to the relation $=_m$.

In order to obtain propositions $\mathfrak{A}$ that are true in a given material process p, we at first introduce the concept of *dialog equivalence* (or *semantical equivalence*), which defines an equivalence relation on the set of propositions.

(2.3) *Definition.* Two propositions $\mathfrak{A}$ and $\mathfrak{B}$ are *dialog equivalent with respect to the material dialog game D_m* ($\mathfrak{A} \equiv_{D_m} \mathfrak{B}$) if and only if (a) there exist $P_{\equiv}(\mathfrak{A})$, $P_{\equiv}(\mathfrak{B})$ for $\mathfrak{A}$ and $\mathfrak{B}$ such that $P_{\equiv}(\mathfrak{A}) = P_{\equiv}(\mathfrak{B})$; (b) $v_{[p]_{\equiv}}(\mathfrak{A}) = v_{[p]_{\equiv}}(\mathfrak{B})$ for all $[p]_{\equiv} \in P_{\equiv}(\mathfrak{A})$.

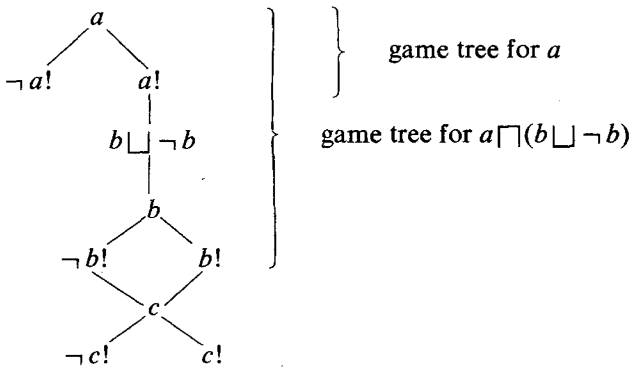
$\mathfrak{A} \equiv_{D_m} \mathfrak{B}$ means that in each dialog in which one of the two propositions $\mathfrak{A}, \mathfrak{B}$ occurs it may be replaced by the other one without thereby influencing the possibilities of winning and losing the dialog.

(2.4) *Definition.* Two propositions $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *value equivalent with respect to their truth values* ($\mathfrak{A} =_{D_m} \mathfrak{B}$) if and only if (a) there exist $P_{=}(\mathfrak{A})$, $P_{=}(\mathfrak{B})$ for $\mathfrak{A}$ and $\mathfrak{B}$ such that $P_{=}(\mathfrak{A}) = P_{=}(\mathfrak{B})$; (b) $v_{[p]_{=}}(\mathfrak{A}) = v_{[p]_{=}}(\mathfrak{B})$ for all $[p]_{=} \in P_{=}(\mathfrak{A})$.

$\mathfrak{A} =_{D_m} \mathfrak{B}$ means that in each dialog in which one of the two propositions is asserted, it may be replaced by the other one, such that the possibilities of winning and losing the subdialogs about $\mathfrak{A}$ and $\mathfrak{B}$ are the same.

For instance we have $a =_{D_m} a \sqcap (b \sqcup \neg b)$ since $\langle a!\rangle =_m \langle a!, (b \sqcup \neg b)!\rangle$, but, in case b is not commensurable with all elementary propositions $\in S_e$, we have $\langle a!\rangle \not\equiv_m \langle a!, (b \sqcup \neg b)!\rangle$ and, therefore, $a \not\equiv_{D_m} a \sqcap (b \sqcap \neg b)$. The proof procedure of the proposition $a \sqcap (b \sqcup \neg b) \sqcap c$

is represented by the following game tree:



where a, b, c are tested by quantum mechanical measurements. The part of the game tree that corresponds to the proposition $a \sqcap (b \sqcup \neg b)$ may be replaced by the game tree for a such that the possibilities of proving these propositions are the same. However, if, after the proof of $a \sqcap (b \sqcup \neg b)$ or the proof of a, a proposition c is tested that is not commensurable with b, the possibilities of proving c in general are different in the two cases.

Although the relation of dialog equivalence depends on the possible material processes $p \in P(\mathfrak{S})$, we can specify some dialog equivalences $\mathfrak{A} \equiv_{D_m} \mathfrak{B}$ purely by means of the formal connective structures of $\mathfrak{A}$ and $\mathfrak{B}$. The dialog rules, together with (2.1), lead to the following.

(2.5) *Formal* properties of the dialog equivalence:
(a) $\equiv_{D_m}$ is an equivalence relation $\subseteq S \times S$
(b) $\mathfrak{A} \sqcap (\mathfrak{B} \sqcap \mathfrak{C}) \equiv_{D_m} (\mathfrak{A} \sqcap \mathfrak{B}) \sqcap \mathfrak{C}$ (associativity of $\sqcap$)
(c) $\mathrm{V} \sqcap \mathfrak{A} \equiv_{D_m} \mathfrak{A} \equiv_{D_m} \mathfrak{A} \sqcap \mathrm{V}$
(d) $\Lambda \sqcap \mathfrak{A} \equiv_{D_m} \Lambda \equiv_{D_m} \mathfrak{A} \sqcap \Lambda$
(e) $\mathfrak{A} \sqcup \mathfrak{B} \equiv_{D_m} \neg(\neg \mathfrak{A} \sqcap \neg \mathfrak{B})$
(f) $\mathfrak{A} \dashv \mathfrak{B} \equiv_{D_m} \neg \mathfrak{A} \sqcup \mathfrak{B}$
(g) $\neg(\neg \mathfrak{A}) \equiv_{D_m} \mathfrak{A}$
(h) $A \wedge B \equiv_{D_m} (A \sqcap B) \sqcap k(A, B) \equiv_{D_m}$ commutations with respect to $\sqcap$
(i) $A \vee B \equiv_{D_m} (A \sqcup B) \sqcup \neg k(A, B) \equiv_{D_m}$ commutations with respect to $\sqcup$
(j) $A \to B \equiv_{D_m} A \dashv (B \sqcap k(A, B)) \equiv_{D_m} A \dashv (k(A, B) \sqcap B)$
(k) $k(A, B) \equiv_{D_m} k(B, A) \equiv_{D_m} k(\neg A, B)$
(l) if $\mathfrak{A} \equiv_{D_m} \mathfrak{B}$ then $\equiv_{D_m} [\mathfrak{A} / \mathfrak{B}]$ (rule of substitution, i.e., if $\mathfrak{B}$ occurs in the $\equiv_{D_m}$ relation, it is replaced by $\mathfrak{A}$)

for all $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in S$ and $A, B \in L$.

Let us now consider the case that the proposition $\neg(\mathfrak{A} \sqcap \mathfrak{B})$ is asserted in a dialog. By means of the argument rules, it can easily be seen that the subdialog about this proposition is won by the proponent if and only if either the subsequent subdialog about $\mathfrak{A}$ is lost by the second participant, or the subdialog about $\mathfrak{A}$ is won and the subsequent subdialog about $\mathfrak{B}$ is lost by the second participant. However, since $\neg(\mathfrak{A} \sqcap \mathfrak{B}) \equiv_{D_m} \neg \mathfrak{A} \sqcup \neg \mathfrak{B}$ is obtained from (2.5), the first case is equivalent to the case where the proponent wins the dialog about $\neg \mathfrak{A}$; the second case is equivalent to the case where the proponent wins the dialog about $\mathfrak{A}$ and the dialog about $\neg \mathfrak{B}$ subsequently.

According to (2.5), propositions that include subpropositions of the form $\neg(\mathfrak{A} \sqcap \mathfrak{B})$ are the only propositions that cannot be reduced to an equivalent conjunction $\sqcap p$. Using now, instead of propositions, variables for propositions, we arrive at the following.

(2.6) The *algorithm* M of *material quantum logic*:

(M1)	$[p\}_{\equiv} \Rightarrow \sqcap d$	if $p \in P(\mathfrak{S})$ and $p \equiv_m d$
(M2.1)	$\mathfrak{A}' \sqcap \neg \mathfrak{A} \sqcap \mathfrak{B}'$	$\Rightarrow \mathfrak{A}' \sqcap \neg(\mathfrak{A} \sqcap \mathfrak{B}) \sqcap \mathfrak{B}'$
(M2.2)	$\mathfrak{A}' \sqcap \mathfrak{A} \sqcap \neg \mathfrak{B} \sqcap \mathfrak{B}'$	$\Rightarrow \mathfrak{A}' \sqcap \neg(\mathfrak{A} \sqcap \mathfrak{B}) \sqcap \mathfrak{B}'$

For simplicity we dispense with the parentheses due to (2.5) (b). The variables $\mathfrak{A}'$ and $\mathfrak{B}'$ include the case that no proposition is inserted.

The algorithm M, together with (2.5), replaces the material dialog game as a proof procedure for compound quantum mechanical propositions. The semantics for material propositions is now comprised in the "semantical" rule (M1), where the dialogic semantics for compound propositions is formalized by the "formal" rules (M2.1), (M2.2) and by (2.5).

We say that $\sqcap p$ *entails* $\mathfrak{A}$ ($\sqcap p \vdash_{\mathrm{M}} \mathfrak{A}$) if and only if $p \in P(\mathfrak{S})$, there exists a $d \in D(\mathfrak{S})$ such that $p \equiv_m d$, and $\mathfrak{A}$ can be deduced from M by means of (2.5) and (2.6). The sets $P(\mathfrak{A})$ can be established by reducing $\mathfrak{A}$ and $\neg \mathfrak{A}$ by means of the inverses of the rules of M, namely,

(M′1) $\sqcap d \Rightarrow [p]_{\equiv}$ if $p \in P(\mathfrak{S})$ and $p \equiv_m d$

(M′2) $\mathfrak{A}' \sqcap \neg(\mathfrak{A} \sqcap \mathfrak{B}) \sqcap \mathfrak{B}' \Rightarrow \mathfrak{A}' \sqcap \neg \mathfrak{A} \sqcap \mathfrak{B}',, \mathfrak{A}' \sqcap \mathfrak{A} \sqcap \neg \mathfrak{B} \sqcap \mathfrak{B}'$

to sets of classes $[p]_{\equiv}$. We have the following.

(2.7) (a) $v_{[p]_{\equiv}}(\mathfrak{A}) = \mathrm{T}$ if and only if $\sqcap p \vdash_{\mathrm{M}} \mathfrak{A}$;
(b) $v_{[p]_{\equiv}}(\mathfrak{A}) = \mathrm{F}$ if and only if $\sqcap p \vdash_{\mathrm{M}} \neg \mathfrak{A}$.

2.2. The Formal Logic

After having established a propositional language of quantum physics and a first step of its formalization by means of the material logic, the program is now to proceed to a logical system that is adequate to the language in the sense of Section 1.1. Since the classical concept of validity is altered in the framework of our semantics, the adequateness means here that the theorems of the logical system exactly are those propositions which are *always true* (see Section 1.3.3). This program is not executed in its fullness here. We restrict our considerations to that part of the program that dispenses with the semantics for material propositions to a certain extent. The reason for this is that the specific Hilbert space interpretation has only been used as an example of a proof procedure for material propositions that is performed *outside* of a dialog, but not as a necessary semantics. Thus it would be interesting to investigate a concept of truth that is widely independent of the underlying semantics for material propositions and that appraises a logical system.

This concept of truth is given by the concept of *formal truth*.

(2.8) *Definition*. A proposition $\mathcal{A}$ is said to be *formally true* if and only if for all substitutions $s_e(\mathcal{A})$ of elementary subpropositions of $\mathcal{A}$ by elementary propositions there exists a substitution $s(s_e(\mathcal{A}))$ of subpropositions of $s_e(\mathcal{A})$ by value equivalent elementary propositions such that $s(s_e(\mathcal{A}))$ is always true, i.e., $v_{[p]_=}(s(s_e(\mathcal{A})))=\mathrm{T}$ for all $[p]_= \in P_=(s(s_e(\mathcal{A})))$.

The formal truth of a proposition $\mathcal{A}$ does not involve the particular interpretation of its elementary propositions but only *formal* semantical properties, which apply to all elementary propositions of S_e. However, in order to establish the formal truth of a proposition, also the semantics for commensurability propositions must be taken into account. We have to investigate whether a commensurability proposition $k(A,B)$ is formally true in the sense that for all substitutions $s_e(A), s_e(B)$ of elementary subpropositions of A and B by elementary propositions there exist substitutions $s(s_e(A)), s(s_e(B))$ of subpropositions of $s_e(A)$ and $s_e(B)$ by value-equivalent elementary propositions such that $k(s(s_e(A)), s(s_e(B)))$ is always true.

The formal truth of commensurability propositions and logically connected propositions has been investigated (Stachow, 1976). As a formal semantical property of elementary propositions, the *idempotence* is assumed, i.e., it is assumed that for each elementary proposition there exists a proof procedure such that, if a proof has been established and the procedure is immediately repeated, the proof is reproduced. In addition, it is assumed that every logically connected proposition is value equivalent to an elementary proposition. For the sake of simplicity, the material dialog game, with the aid of the above assumptions and with the aid of an algorithm for formally true commensurability propositions (both are consistent with the Hilbert space semantics for material propositions), may be replaced by a new dialog game, the *formal dialog game*, which is a proof procedure of the formal truth of logically connected propositions (see the above-cited article). By means of the formal dialog game the formal truth of a proposition A is established if and only if the proponent has a strategy of success within the formal dialog game about A, i.e., he wins the game irrespective of the arguments of the opponent. In previous articles (Stachow, 1976, 1978) logical calculi are investigated that are *complete* and *sound with respect to the dialog-game semantics* in the sense that all those and exactly those propositions can be established to be formally true in the formal dialog game that are theorems in the logical systems. Since the dialog-game semantics establishes the set of admissible valuations of the formal language of quantum physics, as is worked out in Section 1.3, we derive as an immediate consequence that the system of formal quantum logic is *complete* and *sound with respect to the formal language* (see Section 1.1). These results are summarized by means of Diagram 4.

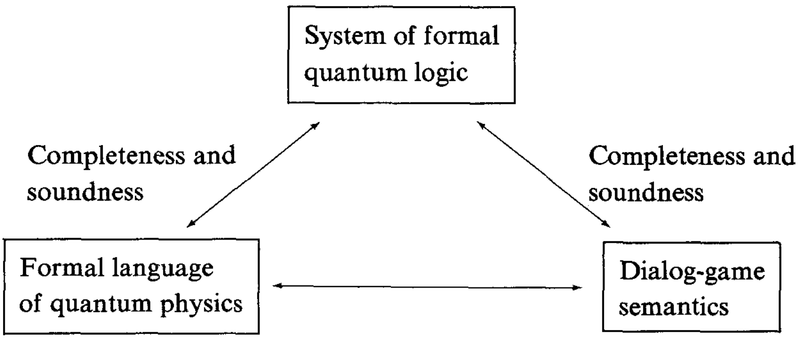


Diagram 4

The above diagram is restricted to the case of *logically* connected propositions due to the results in the previous articles. In the following we extend it to *sequentially* connected propositions. The extension of the formal dialog game to a proof procedure of the formal truth of sequentially connected propositions is straightforward but is not presented here. We proceed directly to the extended system of formal quantum logic which can be demonstrated to be semantically complete and sound with respect to the dialog-game semantics. The formal language of quantum physics is given already in Section 1.3. Thus we obtain the result that the system of formal quantum logic is complete and sound with respect to the formal language.

2.2.1. The Calculus of Quantum Logic Q. In order to establish the system of formal quantum logic we first present the calculus of quantum logic (see Stachow, 1978) which specifies the set of formally true *logically* connected propositions. As a second step the extension of this calculus is given in Section 2.2.2.

The calculus can be written in a simple form if the two particular propositions Λ (the *false proposition*) and V (the *true proposition*) are used. The figures of the calculus are reduced to the form $\{A\}\vdash B$, in the following written as $A \leqslant B$, where A and B are propositions.

(2.9) *Definition.* The *calculus of quantum logic* (Q) is a triple $\langle$QPS$'$, $\leqslant$,RQ$\rangle$, where

(a) QPS$'$ is the Q-propositional syntactic system QPS(1.8), reduced to the set L of logical propositions;

(b) $\leqslant$ is a relation $\subseteq L \times L$ ($A \leqslant B$ is called a *figure*);

(c) RQ is a set consisting of the elements:

(Q1.1) $\Rightarrow A \leqslant A$
(Q1.2) $A \leqslant B,, B \leqslant A \Rightarrow A \leqslant C$
(Q2.1) $\Rightarrow A \wedge B \leqslant A$
(Q2.2) $\Rightarrow A \wedge B \leqslant B$
(Q2.3) $C \leqslant A,, C \leqslant B \Rightarrow C \leqslant A \wedge B$
(Q3.1) $\Rightarrow A \leqslant A \vee B$
(Q3.2) $\Rightarrow B \leqslant A \vee B$
(Q3.3) $A \leqslant C,, B \leqslant C \Rightarrow A \vee B \leqslant C$
(Q4.1) $\Rightarrow A \wedge (A \rightarrow B) \leqslant B$
(Q4.2) $A \wedge C \leqslant B \Rightarrow A \rightarrow C \leqslant A \rightarrow B$
(Q4.3) $A \leqslant B \rightarrow A \Rightarrow B \leqslant A \rightarrow B$
(Q4.4) $B \leqslant A \rightarrow B,, C \leqslant A \rightarrow C \Rightarrow B * C \leqslant A \rightarrow (B * C)$
(Q5.0) $\Rightarrow \Lambda \leqslant A \Rightarrow A \leqslant \mathrm{V}$
(Q5.1) $\Rightarrow A \wedge \neg A \leqslant \Lambda$
(Q5.2) $A \wedge B \leqslant \Lambda \Rightarrow A \rightarrow B \leqslant \neg A$
(Q5.3) $A \leqslant B \rightarrow A \Rightarrow \neg A \leqslant B \rightarrow \neg A$
(Q5.4) $\Rightarrow \mathrm{V} \leqslant A \vee \neg A$

where A, B, C stand for L propositions, and $* \in \{\wedge, \vee, \rightarrow\}$.

If a figure $A \leqslant B$ can be deduced (see Section 1.1) in Q, we write $\vdash_Q A \leqslant B$.

As a result of previous investigations (Stachow, 1978; Mittelstaedt and Stachow, 1978) we have the following.

(2.10) *Theorem.*

(a) A logically connected proposition A is formally true if and only if $\vdash_Q \mathrm{V} \leqslant A$;

(b) a commensurability proposition $k(A,B)$ is formally true if and only if $\vdash_Q \mathrm{V} \leqslant (A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)$.

(2.11) *Definition.* Two propositions $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *formally equivalent* ($\mathfrak{A} =_{D_f} \mathfrak{B}$) if and only if for all substitutions $s_e(\mathfrak{A})$ and $s_e(\mathfrak{B})$ of elementary subpropositions of $\mathfrak{A}$ and $\mathfrak{B}$ by elementary propositions there exist substitutions $s(s_e(\mathfrak{A}))$ and $s(s_e(\mathfrak{B}))$ of subpropositions of $s_e(\mathfrak{A})$ and $s_e(\mathfrak{B})$ by value equivalent elementary propositions such that $s(s_e(\mathfrak{A}))$ and $s(s_e(\mathfrak{B}))$ are value equivalent.

Without giving a proof here, we mention the following result:

(2.12) *Theorem.* (a) $A =_{D_f} B$ if and only if $\vdash_Q A \leqslant B$ and $\vdash_Q B \leqslant A$; (b) $k(A,B) =_{D_f} (A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)$; (c) $A \vee B =_{D_f} \neg(\neg A \wedge \neg B)$, $A \rightarrow B =_{D_f} \neg A \vee (A \wedge B)$, $\neg \Lambda =_{D_f} \mathrm{V}$ for $A, B \in L$.

Because of (a) and (b) the calculus of quantum logic Q is a proof procedure for the formal equivalence of propositions also.

2.2.2. The Calculus of Sequential Quantum Logic SQ. The proof procedures for the formal truth and the formal equivalence is now extended to sequentially connected propositions. In order to establish the calculus of sequential quantum logic we take the following deductive scheme for formal equivalences as a departure which comprises the scheme (2.5) and the result (2.12).

(2.13)
(a) $=_{D_f}$ is an equivalence relation $\subseteq S \times S$
(b) $\mathfrak{A} \sqcap (\mathfrak{B} \sqcap \mathfrak{C}) =_{D_f} (\mathfrak{A} \sqcap \mathfrak{B}) \sqcap \mathfrak{C}$
(c) $\mathfrak{A} \sqcap \mathrm{V} =_{D_f} \mathfrak{A} =_{D_f} \mathrm{V} \sqcap \mathfrak{A}$
(d) $\mathfrak{A} \sqcap \Lambda =_{D_f} \Lambda =_{D_f} \Lambda \sqcap \mathfrak{A}$
(e) $\mathfrak{A} \sqcup \mathfrak{B} =_{D_f} \neg(\neg \mathfrak{A} \sqcap \neg \mathfrak{B})$
(f) $\mathfrak{A} \dashv \mathfrak{B} =_{D_f} \neg \mathfrak{A} \sqcup \mathfrak{B}$
(g) $\neg(\neg \mathfrak{A}) =_{D_f} \mathfrak{A}$
(h) $A \wedge B =_{D_f} (A \sqcap B) \sqcap k(A,B)$
(i) $A \vee B =_{D_f} (A \sqcup B) \sqcup \neg k(A,B)$
(j) $A \rightarrow B =_{D_f} A \dashv (B \sqcap k(A,B))$
(k) $k(A,B) =_{D_f} (A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)$
(l) if $\vdash_Q A \leqslant B$ and $\vdash_Q B \leqslant A$ then $A =_{D_f} B$
(m) if $\mathfrak{A} =_{D_f} \mathfrak{B}$ then $=_{D_f} [\mathfrak{A} / \mathfrak{B}]$ (i.e., $\mathfrak{A}$ is substituted for $\mathfrak{B}$ in $=_{D_f}$)

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ stand for S propositions and A, B stand for L propositions.

This scheme is not yet complete in the sense that it establishes all formal equivalences between sequentially connected propositions. Only an extension of the rule (2.13) (l) which makes use of a calculus of sequential quantum logic completes the scheme. The following calculus incorporates the scheme (2.13) into its rules (SQ5).

(2.14) *Definition*. The *calculus of sequential logic* (SQ) is a triple $\langle \text{QPS}, \leqslant, \text{RSQ} \rangle$, where

(a) QPS is the Q-propositional syntactic system (1.8);

(b) $\leqslant$ is a relation $\subseteq S \times S$ (called *sequential implication*, $\mathcal{A} \leqslant \mathcal{B}$ is called a *figure*);

(c) RSQ is a set consisting of the elements (Q1.1)–(Q5.4) and the additional elements:

(SQ1) $$A \leqslant B \Rightarrow \mathcal{A} \sqcap A \leqslant B$$

(SQ2.1)

$$\mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \mathcal{A} \leqslant \neg(\neg \mathcal{B} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1 \sqcap \neg \mathcal{C}),,$$
$$\mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap \mathcal{A} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1 \leqslant \mathcal{C}$$
$$\Rightarrow \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1 \leqslant \mathcal{C}$$

(SQ2.2)

$$\mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \neg \mathcal{A} \leqslant \neg(\mathcal{B} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1 \sqcap \neg \mathcal{C}),,$$
$$\mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap \mathcal{B} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1 \leqslant \mathcal{C}$$
$$\Rightarrow \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1 \leqslant \mathcal{C}$$

(SQ3.1)

$$\mathcal{C} \sqcap \mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \mathcal{A} \leqslant \neg(\neg \mathcal{B} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1),,$$
$$\mathcal{C} \leqslant \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap \mathcal{A} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$$
$$\Rightarrow \mathcal{C} \leqslant \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$$

(SQ3.2)

$$\mathcal{C} \sqcap \mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \neg \mathcal{A} \leqslant \neg(\mathcal{B} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1),,$$
$$\mathcal{C} \leqslant \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \neg(\mathcal{A}_n \sqcap \mathcal{B} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$$
$$\Rightarrow \mathcal{C} \leqslant \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$$

(SQ4) $$\mathcal{C} \sqcap \mathcal{A} \leqslant \Lambda \Rightarrow \mathcal{C} \leqslant \neg \mathcal{A}$$

(SQ5.1) $$\leqslant [\mathcal{A} \sqcap (\mathcal{B} \sqcap \mathcal{C})] \Leftrightarrow \leqslant [(\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{C}]$$

(SQ5.2) $$\leqslant [\mathcal{A} \sqcap \mathrm{V}] \Leftrightarrow \leqslant [\mathcal{A}] \Leftrightarrow \leqslant [\mathrm{V} \sqcap \mathcal{A}]$$

(SQ5.3) $$\leqslant [\mathcal{A} \sqcap \Lambda] \Leftrightarrow \leqslant [\Lambda] \Leftrightarrow \leqslant [\Lambda \sqcap \mathcal{A}]$$

(SQ5.4) $$\leqslant [\mathcal{A} \sqcup \mathcal{B}] \Leftrightarrow \leqslant [\neg(\neg \mathcal{A} \sqcap \neg \mathcal{B})]$$

(SQ5.5) $$\leqslant [\mathcal{A} \vdash \mathcal{B}] \Leftrightarrow \leqslant [\neg \mathcal{A} \sqcup \mathcal{B}]$$

(SQ5.6) $\leqslant[\neg(\neg\mathcal{A})]\Leftrightarrow\leqslant[\mathcal{A}]$

(SQ5.7) $\leqslant[A\wedge B]\Leftrightarrow\leqslant[(A\sqcap B)\sqcap k(A,B)]$

(SQ5.8) $\leqslant[A\vee B]\Leftrightarrow\leqslant[(A\sqcup B)\sqcup\neg k(A,B)]$

(SQ5.9) $\leqslant[A\rightarrow B]\Leftrightarrow\leqslant[A\dashv(B\sqcap k(A,B))]$

(SQ5.10)

$$\leqslant[k(A,B)]\Leftrightarrow\leqslant[(A\wedge B)\vee(A\wedge\neg B)\vee(\neg A\wedge B)\vee(\neg A\wedge\neg B)]$$

(SQ5.11) $A\leqslant B,, B\leqslant A,, \leqslant[A]\Rightarrow\leqslant[B]$

$\leqslant[\mathcal{A}]$ designates that a figure contains the proposition $\mathcal{A}$; the place in which $\mathcal{A}$ occurs is the same for both of the sides of a rule. $\mathcal{A},\mathcal{B},\mathcal{C}$ stand for S propositions; $\mathcal{A}_1,\ldots,\mathcal{A}_n,\mathcal{B}_1,\ldots,\mathcal{B}_n$ $(n\in\mathbb{N})$ stand for S propositions or no proposition; A,B stand for L propositions.

If a figure $\mathcal{A}\leqslant\mathcal{B}$ can be deduced in the calculus SQ we write $\vdash_{\mathrm{SQ}}\mathcal{A}\leqslant\mathcal{B}$.

By means of the calculus SQ the system of formal quantum logic can now be established:

(2.15) *Definition*. The *system of formal quantum logic* (QLS) is a triple $\langle \mathrm{QPS}, \leqslant, \mathrm{Th}\rangle$, where

(a) QPS is the Q-propositional syntactic system (1.8);
(b) $\leqslant$ is the sequential implication $\subseteq S\times S$;
(c) Th is the set of theorems $\{\mathcal{A}\in S:\vdash_{\mathrm{SQ}}\mathrm{V}\leqslant\mathcal{A}\}$.

The main result of our consideration of the formal quantum logic is formulated by the following theorem. The proof, which involves the details of the formal dialog game about sequentially connected propositions, will be given elsewhere.

(2.16) *Theorem* (*Semantical completeness* and *soundness* of QLS). $\mathcal{A}\in$ Th if and only if the formal truth of $\mathcal{A}$ can be established in the formal dialog game.

Since the dialogic semantics establishes the formal language of quantum physics (QL), we have as a consequence of (2.16) the following.

(2.17) *Completeness* and *soundness* of QLS with respect to QL. $\mathcal{A}\in\mathrm{Th}$ if and only if for all substitutions $s_e(\mathcal{A})$ of elementary propositions of $\mathcal{A}$ by elementary propositions there exists a substitution $s(s_e(\mathcal{A}))$ of subpropositions of $s_e(\mathcal{A})$ by value equivalent elementary propositions such that $v_{[p]_=}(s(s_e(\mathcal{A})))=\mathrm{T}$ for all $[p]_=\in P_=(s(s_e(\mathcal{A})))$.

We can now complete the deductive scheme for formal equivalences (2.13) in order to establish all formal equivalences between sequential propositions. The rule (2.13) (l) is replaced by the following rules for formal equivalences which are already incorporated in the rules (SQ2) and (SQ3) of the calculus:

$$(2.13)\ (11)\quad \vdash_{SQ}\mathfrak{A}_1\sqcap\cdots\sqcap\mathfrak{A}_n\sqcap\mathfrak{A}\leqslant\neg(\neg\mathfrak{B}\sqcap\mathfrak{B}_n\sqcap\cdots\sqcap\mathfrak{B}_1)$$
$$\rightsquigarrow\quad \mathfrak{A}_1\sqcap\neg(\mathfrak{A}_2\sqcap\neg(\cdots\sqcap\neg(\mathfrak{A}_n\sqcap\mathfrak{A}\sqcap\mathfrak{B}_n)\sqcap\cdots)\sqcap\mathfrak{B}_2)\sqcap\mathfrak{B}_1$$
$$=_{D_f}\mathfrak{A}_1\sqcap\neg(\mathfrak{A}_2\sqcap\neg(\cdots\sqcap\neg(\mathfrak{A}_n\sqcap(\mathfrak{A}\sqcap\mathfrak{B})\sqcap\mathfrak{B}_n)\sqcap\cdots)\sqcap\mathfrak{B}_2)\sqcap\mathfrak{B}_1$$
$$(12)\quad \vdash_{SQ}\mathfrak{A}_1\sqcap\cdots\sqcap\mathfrak{A}_n\sqcap\neg\mathfrak{A}\leqslant\neg(\mathfrak{B}\sqcap\mathfrak{B}_n\sqcap\cdots\sqcap\mathfrak{B}_1)$$
$$\rightsquigarrow\quad \mathfrak{A}_1\sqcap\neg(\mathfrak{A}_2\sqcap\neg(\cdots\sqcap\neg(\mathfrak{A}_n\sqcap\mathfrak{B}\sqcap\mathfrak{B}_n)\sqcap\cdots)\sqcap\mathfrak{B}_2)\sqcap\mathfrak{B}_1$$
$$=_{D_f}\mathfrak{A}_1\sqcap\neg(\mathfrak{A}_2\sqcap\neg(\cdots\sqcap\neg(\mathfrak{A}_n\sqcap(\mathfrak{A}\sqcap\mathfrak{B})\sqcap\mathfrak{B}_n)\sqcap\cdots)\sqcap\mathfrak{B}_2)\sqcap\mathfrak{B}_1$$

Then one can show that two propositions $\mathfrak{A}$ and $\mathfrak{B}$ are formally equivalent if and only if $\mathfrak{A}=_{D_f}\mathfrak{B}$ is deducible within the scheme (2.13) extended to (11) and (12).

The rule (2.13) (l):

$$\vdash_Q A\leqslant B\quad\text{and}\quad\vdash_Q B\leqslant A\quad\leftrightsquigarrow\quad A=_{D_f}B\qquad\text{with } A,B\in L$$

can be deduced within the extended scheme. Therefore the formal equivalences between logically connected propositions can be established by means of (2.13). On the other hand, the restriction of the rules (2.13) (11) and (12) to L-propositions:

$$\vdash_Q A\leqslant B\rightsquigarrow\mathfrak{A}_1\sqcap\neg(\mathfrak{A}_2\sqcap\neg(\cdots\sqcap\neg(\mathfrak{A}_n\sqcap A\sqcap\mathfrak{B}_n)\sqcap\cdots)\sqcap\mathfrak{B}_2)\sqcap\mathfrak{B}_1$$
$$=_{D_f}\mathfrak{A}_1\sqcap\neg(\mathfrak{A}_2\sqcap\neg(\cdots\sqcap\neg(\mathfrak{A}_n\sqcap(A\sqcap B)\sqcap\mathfrak{B}_n)\sqcap\cdots)\sqcap\mathfrak{B}_2)\sqcap\mathfrak{B}_1$$
$$\vdash_Q\neg A\leqslant\neg B\rightsquigarrow\mathfrak{A}_1\sqcap\neg(\mathfrak{A}_2\sqcap\neg(\cdots\sqcap\neg(\mathfrak{A}_n\sqcap B\sqcap\mathfrak{B}_n)\sqcap\cdots)\sqcap\mathfrak{B}_2)\sqcap\mathfrak{B}_1$$
$$=_{D_f}\mathfrak{A}_1\sqcap\neg(\mathfrak{A}_2\sqcap\neg(\cdots\sqcap\neg(\mathfrak{A}_n\sqcap(A\sqcap B)\sqcap\mathfrak{B}_n)\sqcap\cdots)\sqcap\mathfrak{B}_2)\sqcap\mathfrak{B}_1$$

are deducible within (2.13). Moreover, these rules are equivalent to the rule (2.13) (l) within the scheme (2.13). The extension of the rule (2.13) (l) to sequential propositions:

$$\vdash_{SQ}\mathfrak{A} \leqslant \mathfrak{B} \quad \text{and} \quad \vdash_{SQ}\mathfrak{B} \leqslant \mathfrak{A} \;\leftrightharpoons\; \mathfrak{A} =_{D_f} \mathfrak{B} \qquad \text{with } \mathfrak{A}, \mathfrak{B} \in S$$

is not valid, as can easily be seen by means of the following two counterexamples for both of the directions: We have $\vdash_{SQ}(A \sqcup \mathfrak{B}) \sqcap A \leqslant (A \sqcup \mathfrak{C}) \sqcap A$ and $\vdash_{SQ}(A \sqcup \mathfrak{C}) \sqcap A \leqslant (A \sqcup \mathfrak{B}) \sqcap A$, but not $(A \sqcup \mathfrak{B}) \sqcap A =_{D_f} (A \sqcup \mathfrak{C}) \sqcap A$. On the other hand, we have $\mathfrak{A} =_{D_f} \mathfrak{A}$ but not $\vdash_{SQ}\mathfrak{A} \leqslant \mathfrak{A}$ for $\mathfrak{A} \in S$.

The inverse of the rules (2.13) (11) and (12) can be shown to be valid.

2.2.3. Properties of Sequential Quantum Logic. For the following consideration of the algebraic representation of quantum logic, some further properties of quantum logic are important.

(2.18) Some properties of SQ:

(a) The inverse of the rule (SQ4) is valid, i.e.,

$$\vdash_{SQ}\mathfrak{C} \leqslant \neg\mathfrak{A} \;\rightharpoondown\; \vdash_{SQ}\mathfrak{C} \sqcap \mathfrak{A} \leqslant \Lambda;$$

(b) the rules, which are generated by the rules (SQ2) and (SQ3) by interchanging the second premise and the conclusion, are valid;

(c) $\vdash_{SQ}\mathfrak{A} \leqslant \Lambda \;\rightharpoondown\; \vdash_{SQ}\mathfrak{A} \leqslant \mathfrak{B}$;

(d) $\vdash_{SQ}\mathfrak{B} \leqslant \mathfrak{C} \;\rightharpoondown\; \vdash_{SQ}\mathfrak{A} \sqcap \mathfrak{B} \leqslant \mathfrak{C}$;

(e) $\vdash_{SQ}\mathfrak{A} \sqcap \mathfrak{B} \leqslant \mathfrak{C} \;\leftrightharpoons\; \vdash_{SQ}\mathfrak{A} \leqslant \mathfrak{B} \dashv \mathfrak{C}$;

(f) transitivity of the sequential implication:

$$\vdash_{SQ}\mathfrak{A} \leqslant \mathfrak{B} \quad \text{and} \quad \vdash_{SQ}\mathfrak{B} \leqslant \mathfrak{C} \;\rightharpoondown\; \vdash_{SQ}\mathfrak{A} \leqslant \mathfrak{C};$$

(g) consistency of the sequential implication with respect to the formal equivalence:

$$\mathfrak{A} =_{D_f} \mathfrak{B} \quad \text{and} \quad \vdash_{SQ} \leqslant [\mathfrak{A}] \;\rightharpoondown\; \vdash_{SQ} \leqslant [\mathfrak{B}].$$

Remarks on the proofs: (a) can be proved by means of induction on the length of the deduction of the premise $\mathfrak{C} \leqslant \neg\mathfrak{A}$ by means of the rules (SQ): If $\mathfrak{C} \leqslant \neg\mathfrak{A}$ is the premise of the rule (SQ1), i.e., $\mathfrak{C}, \mathfrak{A} \in L$, $\vdash_{SQ}\mathfrak{C} \sqcap \mathfrak{A} \leqslant \Lambda$ follows by means of the rules (Q). If $\mathfrak{C}' \leqslant \neg\mathfrak{A}'$ is a predecessor of $\mathfrak{C} \leqslant \neg\mathfrak{A}$ within the deduction of $\mathfrak{C} \leqslant \neg\mathfrak{A}$, it follows by means of the inductive hypothesis $\mathfrak{C}' \sqcap \mathfrak{A}' \leqslant \Lambda$ that $\mathfrak{C} \sqcap \mathfrak{A} \leqslant \Lambda$ also is deducible. For instance if $\neg\mathfrak{A} = \neg(\mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap (\tilde{\mathfrak{A}} \sqcup \tilde{\mathfrak{B}}) \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2)$, and the rule (SQ3.1) is applied, the predecessors are $\mathfrak{C} \sqcap \mathfrak{A}_2 \sqcap \cdots$

$\sqcap \mathfrak{A}_n \sqcap \tilde{\mathfrak{A}} \leqslant \neg(\neg \tilde{\mathfrak{B}} \sqcap \mathfrak{B}_n \sqcap \cdots \sqcap \mathfrak{B}_2)$, $\mathcal{C} \leqslant \neg(\mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg (\mathfrak{A}_n \sqcap \mathfrak{A} \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2)$. The inductive hypothesis yields $\vdash_{SQ} \mathcal{C} \sqcap \mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap \mathfrak{A} \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2 \leqslant \Lambda$. Together with the first predecessor and by means of the rule (SQ2.1), $\vdash_{SQ} \mathcal{C} \sqcap \mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap (\tilde{\mathfrak{A}} \sqcap \tilde{\mathfrak{B}}) \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2 \leqslant \Lambda$ follows.

(b) We prove the rule:

$$\vdash_{SQ} \mathfrak{A}_1 \sqcap \cdots \sqcap \mathfrak{A}_n \sqcap \mathfrak{A} \leqslant \neg(\neg \mathfrak{B} \sqcap \mathfrak{B}_n \sqcap \cdots \sqcap \mathfrak{B}_1 \sqcap \neg \mathcal{C})$$

and

$$\vdash_{SQ} \mathfrak{A}_1 \sqcap \neg(\mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap (\mathfrak{A} \sqcap \mathfrak{B}) \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2) \sqcap \mathfrak{B}_1 \leqslant \mathcal{C}$$
$$\rightharpoonup \vdash_{SQ} \mathfrak{A}_1 \sqcap \neg(\mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap \mathfrak{A} \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2) \sqcap \mathfrak{B}_1 \leqslant \mathcal{C}.$$

The other rules can be proved analogously. By means of (a) and the rule (SQ4) it follows from the first premise that $\vdash_{SQ} \mathfrak{A}_1 \sqcap \cdots \sqcap \mathfrak{A}_n \sqcap \mathfrak{A} \sqcap \neg \mathfrak{B} \leqslant \neg(\mathfrak{B}_n \sqcap \cdots \sqcap \mathfrak{B}_1 \sqcap \neg \mathcal{C})$. By means of (SQ5.2) and (SQ5.11) we have $\vdash_{SQ} \mathfrak{A}_1 \sqcap \cdots \sqcap \mathfrak{A}_n \sqcap \mathfrak{A} \sqcap \neg \mathfrak{B} \leqslant \neg(\neg \Lambda \sqcap \mathfrak{B}_n \sqcap \cdots \sqcap \mathfrak{B}_1 \sqcap \neg \mathcal{C})$. Together with the second premise and by means of the rules (SQ5.6) and (SQ2.1) $\vdash_{SQ} \mathfrak{A}_1 \sqcap \neg(\mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap (\mathfrak{A} \sqcap \neg(\neg \mathfrak{B} \sqcap \Lambda)) \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2) \sqcap \mathfrak{B}_1 \leqslant \mathcal{C}$ and by means of (SQ5.2) and (SQ5.3) finally $\vdash_{SQ} \mathfrak{A}_1 \sqcap \neg(\mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap \mathfrak{A} \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2) \sqcap \mathfrak{B}_1 \leqslant \mathcal{C}$ follows.

(c), (d) can be proved by means of induction on the length of the deduction of the premises, analogously to the proof of (a).

(e) From $\vdash_{SQ} \mathfrak{A} \sqcap \mathfrak{B} \leqslant \mathcal{C}$ it follows by means of (a) that $\vdash_{SQ} \mathfrak{A} \sqcap \mathfrak{B} \sqcap \neg \mathcal{C} \leqslant \Lambda$, and by means of (SQ4) that $\vdash_{SQ} \mathfrak{A} \leqslant \neg(\mathfrak{B} \sqcap \neg \mathcal{C})$. By means of (SQ5.4) and (SQ5.5) we obtain $\vdash_{SQ} \mathfrak{A} \leqslant \mathfrak{B} \vdash \mathcal{C}$. Since the rules used in this proof are reversible it also follows from $\vdash_{SQ} \mathfrak{A} \leqslant \mathfrak{B} \dashv \mathcal{C}$ that $\vdash_{SQ} \mathfrak{A} \sqcap \mathfrak{B} \leqslant \mathcal{C}$.

(f) From $\vdash_{SQ} \mathfrak{B} \leqslant \mathcal{C}$ it follows by means of (d) that $\vdash_{SQ} \mathfrak{A} \sqcap \mathfrak{B} \leqslant \mathcal{C}$ also. By means of (SQ2.1) we obtain from $\vdash_{SQ} \mathfrak{A} \leqslant \mathfrak{B}$ and $\vdash_{SQ} \mathfrak{A} \sqcap \mathfrak{B} \leqslant \mathcal{C}$ the conclusion $\vdash_{SQ} \mathfrak{A} \leqslant \mathcal{C}$.

(g) can be proved by means of induction on the length of the deduction of $\mathfrak{A} =_{D_f} \mathfrak{B}$ within the scheme (2.13). If $\mathfrak{A} =_{D_f} \mathfrak{B}$ is one of the formal equivalences (2.13) (b)–(k), the conclusion is obtained by means of the corresponding rule of SQ. If $\mathfrak{A}' =_{D_f} \mathfrak{B}'$ is a predecessor of $\mathfrak{A} =_{D_f} \mathfrak{B}$ within the deduction of $\mathfrak{A} =_{D_f} \mathfrak{B}$, i.e., one of the rules (2.13) (a) and (m) is applied, it follows by means of the inductive hypothesis and the rules of SQ that $\vdash_{SQ} \leqslant [\mathfrak{B}]$. If one of the rules (2.13) (l1) and (l2) is applied, $\vdash_{SQ} \leqslant [\mathfrak{B}]$ is obtained by means of the premise of this rule. For instance, if $\mathfrak{B} = \mathfrak{A}_1 \sqcap \neg(\mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap (\tilde{\mathfrak{A}} \sqcap \tilde{\mathfrak{B}}) \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2) \sqcap \mathfrak{B}_1$,

$\mathcal{A} = \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap \tilde{\mathcal{A}} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$ and the rule (2.13) (11) is applied, we have $\vdash_{SQ} \mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \tilde{\mathcal{A}} \leqslant \neg(\neg \tilde{\mathcal{B}} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1)$ as the premise. If $\mathcal{A}$ occurs on the right side of $\leqslant$ in $\leqslant[\mathcal{A}]$, the figure $\leqslant[\mathcal{A}]$ has the form $\mathcal{C} \leqslant \mathcal{C}_1 \sqcap \neg(\mathcal{C}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{C}_m \sqcap \mathcal{A} \sqcap \tilde{\mathcal{C}}_m) \sqcap \cdots) \sqcap \tilde{\mathcal{C}}_2) \sqcap \tilde{\mathcal{C}}_1$. By means of the rules (a), (c), (d) and (SQ4) it follows from the premise that $\vdash_{SQ} \mathcal{C} \sqcap \mathcal{C}_1 \sqcap \cdots \sqcap \mathcal{C}_m \sqcap \mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \tilde{\mathcal{A}} \leqslant \neg(\neg \tilde{\mathcal{B}} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1 \sqcap \tilde{\mathcal{C}}_m \sqcap \cdots \sqcap \tilde{\mathcal{C}}_1)$. Therefore we obtain from (SQ3.1) that $\vdash_{SQ} \mathcal{C} \leqslant \mathcal{C}_1 \sqcap \neg(\mathcal{C}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{C}_m \sqcap \mathcal{B} \sqcap \tilde{\mathcal{C}}_m) \sqcap \cdots) \sqcap \tilde{\mathcal{C}}_2) \sqcap \tilde{\mathcal{C}}_1$. If $\mathcal{A}$ occurs on the left side of $\leqslant$ in $\leqslant[\mathcal{A}]$, the figure $\leqslant[\mathcal{A}]$ has the form $\mathcal{C}_1 \sqcap \neg(\mathcal{C}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{C}_m \sqcap \mathcal{A} \sqcap \tilde{\mathcal{C}}_m) \sqcap \cdots) \sqcap \tilde{\mathcal{C}}_2) \sqcap \tilde{\mathcal{C}}_1 \leqslant \mathcal{C}$. It follows from the premise that

$$\vdash_{SQ} \mathcal{C}_1 \sqcap \cdots \sqcap \mathcal{C}_m \sqcap \mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \tilde{\mathcal{A}}$$
$$\leqslant \neg(\neg \tilde{\mathcal{B}} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1 \sqcap \tilde{\mathcal{C}}_m \sqcap \cdots \sqcap \tilde{\mathcal{C}}_1 \sqcap \neg \mathcal{C})$$

Then it follows from (SQ2.1) that

$$\vdash_{SQ} \mathcal{C}_1 \sqcap \neg(\mathcal{C}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{C}_m \sqcap \mathcal{B} \sqcap \tilde{\mathcal{C}}_m) \sqcap \cdots) \sqcap \tilde{\mathcal{C}}_2) \sqcap \tilde{\mathcal{C}}_2 \leqslant \mathcal{C}.$$

In order to establish further properties of the sequential quantum logic, it is useful to introduce a new operation (*) on S which inverts the sequence of the subpropositions of a sequential proposition. This operation is recursively given by the following definition.

(2.19) *Definition.* The operation * is a mapping $S \to S$, $\mathcal{A} \mapsto \mathcal{A}^*$, where
(a) $a^* = a$ for $a \in S_e$;
(b) $(\mathcal{A} \sqcap \mathcal{B})^* = \mathcal{B}^* \sqcap \mathcal{A}^*$;
(c) $(\neg \mathcal{A})^* = \neg \mathcal{A}^*$.

Obviously we have $A^* = A$ for $A \in L$ because of the commutativity of "$\sqcap$" within logically connected propositions. Since $(\mathcal{A}^*)^* = \mathcal{A}$, this operation reminds one of an involution.

(2.20) Properties of sequential quantum logic:
(a) $\vdash_{SQ} \mathcal{A} \leqslant \mathcal{B} \rightharpoondown \vdash_{SQ} \neg \mathcal{B}^* \leqslant \neg \mathcal{A}^*$;
(b) $\mathcal{A} =_{D_f} \mathcal{B} \rightharpoondown \mathcal{A}^* =_{D_f} \mathcal{B}^*$.

Remarks. (a) can be considered as a generalization of the rule

$$\vdash_Q A \leqslant B \rightharpoondown \vdash_Q \neg B \leqslant \neg A \qquad \text{with } A, B \in L$$

in sequential quantum logic. (b) determines that the relation of formal equivalence of two propositions is preserved if the sequence of the subpropositions of the two propositions is inverted.

Remarks on the Proofs. (a) can be proved by means of induction on the length of the deduction of the premise $\mathcal{A} \leqslant \mathcal{B}$ within the calculus SQ. If $\mathcal{A} \leqslant \mathcal{B}$ is the premise of the rule (SQ1), then $\mathcal{A}, \mathcal{B} \in L$ and $\vdash_{SQ} \neg \mathcal{B}^* \leqslant \neg \mathcal{A}^*$ follows. If $\mathcal{A}' \leqslant \mathcal{B}'$ is a predecessor of $\mathcal{A} \leqslant \mathcal{B}$ within the deduction of $\mathcal{A} \leqslant \mathcal{B}$, the inductive hypothesis consists in $\vdash_{SQ} \neg \mathcal{B}'^* \leqslant \neg \mathcal{A}'^*$. Then one has to show that $\vdash_{SQ} \neg \mathcal{B}^* \leqslant \neg \mathcal{A}^*$ also. If $\mathcal{A} \leqslant \mathcal{B}$ has the form $\tilde{\mathcal{A}} \sqcap A \leqslant B$ and the rule (SQ1) is applied, we have $\vdash_{SQ} \neg B \leqslant \neg A$, and by means of (2.18) (a) we have $\vdash_{SQ} \neg B \sqcap A \leqslant \Lambda$. From (2.18) (c) and (a) it follows that $\vdash_{SQ} \neg B \sqcap A \sqcap \tilde{\mathcal{A}}^* \leqslant \Lambda$. By means of (SQ4) we obtain $\vdash_{SQ} \neg B \leqslant \neg(A \sqcap \tilde{\mathcal{A}})^*$. If $\mathcal{A} = \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\tilde{\mathcal{A}} \sqcap \tilde{\mathcal{B}}) \sqcap \mathcal{B}_n \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$ and the rule (SQ2.1) is applied, we have by means of the inductive hypothesis $\vdash_{SQ} \neg \mathcal{B}^* \sqcap \mathcal{B}_1^* \sqcap \cdots \sqcap \mathcal{B}_n^* \sqcap \neg \tilde{\mathcal{B}}^* \leqslant \neg(\tilde{\mathcal{A}}^* \sqcap \mathcal{A}_n^* \sqcap \cdots \sqcap \mathcal{A}_1^*)$ and $\vdash_{SQ} \neg \mathcal{B}^* \leqslant \neg(\mathcal{B}_1^* \sqcap \neg(\mathcal{B}_2^* \sqcap \neg(\cdots \sqcap \neg(\mathcal{B}_n^* \sqcap \tilde{\mathcal{A}}^* \sqcap \mathcal{A}_n^*) \sqcap \cdots) \sqcap \mathcal{A}_2^*) \sqcap \mathcal{A}_1^*$. By means of (SQ3.2) we obtain $\vdash_{SQ} \neg \mathcal{B}^* \leqslant \neg \mathcal{A}^*$. The cases in which the rules (SQ2.2) to (SQ3.2) are applied are analogous. If $\mathcal{A} \leqslant \mathcal{B}$ has the form $\mathcal{A} \leqslant \neg \mathcal{C}$ and the rule (SQ4) is applied, we have by means of the inductive hypothesis $\vdash_{SQ} \neg \Lambda \leqslant \neg(\mathcal{C}^* \sqcap \mathcal{A}^*)$. By means of (2.18) (a) it follows that $\vdash_{SQ} \neg \Lambda \sqcap \mathcal{C}^* \sqcap \mathcal{A}^* \leqslant \Lambda$, and by means of the rules (SQ4), (SQ5.2), and (SQ5.6) we have $\vdash_{SQ} \neg(\neg \mathcal{C})^* \leqslant \neg \mathcal{A}^*$. It can easily be seen that in the case that one of the rules (SQ5) is applied the conclusion $\vdash_{SQ} \neg \mathcal{B}^* \leqslant \neg \mathcal{A}^*$ also follows from the inductive hypothesis.

(b) can be proved by means of induction on the length of the deduction of the premise $\mathcal{A} =_{D_f} \mathcal{B}$ within the scheme (2.13). If $\mathcal{A} =_{D_f} \mathcal{B}$ is one of the formal equivalences (2.13) (b)–(k), the conclusion $\mathcal{A}^* =_{D_f} \mathcal{B}^*$ is easily obtained. If $\mathcal{A}' =_{D_f} \mathcal{B}'$ is a predecessor of $\mathcal{A} =_{D_f} \mathcal{B}$ within the deduction of $\mathcal{A} =_{D_f} \mathcal{B}$, i.e., one of the rules (2.13) (a) and (m) is applied, we have $\mathcal{A}'^* =_{D_f} \mathcal{B}'^*$ by means of the inductive hypothesis. Then it can easily be shown that $\mathcal{A}^* =_{D_f} \mathcal{B}^*$ also. If $\mathcal{A} = \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \neg(\mathcal{A}_n \sqcap \tilde{\mathcal{A}} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$, $\mathcal{B} = \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\tilde{\mathcal{A}} \sqcap \tilde{\mathcal{B}}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$ and the rule (2.13) (11) is applied, we have $\vdash_{SQ} \mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \tilde{\mathcal{A}} \leqslant \neg(\neg \tilde{\mathcal{B}} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1)$. By means of (a) it follows that $\vdash_{SQ} \mathcal{B}_1^* \sqcap \cdots \sqcap \mathcal{B}_n^* \sqcap \neg \tilde{\mathcal{B}}^* \leqslant \neg(\tilde{\mathcal{A}}^* \sqcap \mathcal{A}_n^* \sqcap \cdots \sqcap \mathcal{A}_1^*)$ also. Therefore we have by means of (2.13) (12) $\mathcal{B}_1^* \sqcap \neg(\mathcal{B}_2^* \sqcap \neg(\cdots \sqcap \neg(\mathcal{B}_n^* \sqcap \tilde{\mathcal{A}}^* \sqcap \mathcal{A}_n^*) \sqcap \cdots) \sqcap \mathcal{A}_2^*) \sqcap \mathcal{A}_1^* =_{D_f} \mathcal{B}_1^* \sqcap \neg(\mathcal{B}_2^* \sqcap \neg(\cdots \sqcap \neg(\mathcal{B}_n^* \sqcap (\tilde{\mathcal{B}}^* \sqcap \tilde{\mathcal{A}}^*) \sqcap \mathcal{A}_n^*) \sqcap \cdots) \sqcap \mathcal{A}_2^*) \sqcap \mathcal{A}_1^*$, thus $\mathcal{A}^* =_{D_f} \mathcal{B}^*$. The case in which the rule (2.13) (12) is applied, is analogous.

(2.21) *Theorem.* For $A, B \in L$ and $\mathcal{C} \in S$:

(a) (i) $\vdash_{SQ} A \vee (\neg A \wedge B) \leqslant A \sqcap B$;

(ii) $\vdash_{SQ} \mathcal{C} \leqslant A \sqcap B \rightharpoondown \vdash_{SQ} \mathcal{C} \leqslant A \vee (\neg A \wedge B)$;

(iii) $\vdash_{SQ} A \sqcup B \leqslant A \vee B$;

(iv) $\vdash_{SQ} A \sqcup B \leqslant \mathcal{C} \rightharpoondown \vdash_{SQ} A \vee B \leqslant \mathcal{C}$.

(b) (i) $\vdash_{\mathrm{SQ}} A \wedge B \leqslant A \sqcap B$;

(ii) $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A \sqcap B \rightsquigarrow \vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A \wedge B$;

(iii) $\mathrm{A}_{\mathrm{SQ}} A \sqcap B \leqslant (A \vee \neg B) \wedge B$;

(iv) $\vdash_{\mathrm{SQ}} A \sqcap B \leqslant \mathcal{C} \rightsquigarrow \vdash_{\mathrm{SQ}} (A \vee \neg B) \wedge B \leqslant \mathcal{C}$.

(c) Let $S_{\sqcap}$ be the set of all propositions $\sqcap_{i=1}^{n} A_i$ which are sequential conjunctions of logical propositions $A_i \in L$; let $'$: $S_{\sqcap} \to L$ be a mapping, recursively defined by (α) $A' = A$ for $A \in L$; (β) $(\sqcap_{i=1}^{n} A_i)' = ((\sqcap_{i=1}^{n-1} A_i)' \vee \neg A_n) \wedge A_n$.

(i) $\vdash_{\mathrm{SQ}} \wedge_{i=1}^{n} A_i \leqslant \sqcap_{i=1}^{n} A_i$;

(ii) $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant \sqcap_{i=1}^{n} A_i \rightsquigarrow \vdash_{\mathrm{SQ}} \mathcal{C} \leqslant \wedge_{i=1}^{n} A_i$;

(iii) $\vdash_{\mathrm{SQ}} \sqcap_{i=1}^{n} A_i \leqslant (\sqcap_{i=1}^{n} A_i)'$;

(iv) $\vdash_{\mathrm{SQ}} \sqcap_{i=1}^{n} A_i \leqslant \mathcal{C} \rightsquigarrow \vdash_{\mathrm{SQ}} (\sqcap_{i=1}^{n} A_i)' \leqslant \mathcal{C}$.

(d) For any sequential proposition $\mathcal{A} \in S$ there exist logical propositions $\bar{A}, \tilde{A} \in L$ such that

(i) $\vdash_{\mathrm{SQ}} \bar{A} \leqslant \mathcal{A}$;

(ii) $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant \mathcal{A} \rightsquigarrow \vdash_{\mathrm{SQ}} \mathcal{C} \leqslant \bar{A}$;

(iii) $\vdash_{\mathrm{SQ}} \mathcal{A} \leqslant \tilde{A}$;

(iv) $\vdash_{\mathrm{SQ}} \mathcal{A} \leqslant \mathcal{C} \rightsquigarrow \vdash_{\mathrm{SQ}} \tilde{A} \leqslant \mathcal{C}$.

Remark. Since (a), (b), (c) are useful subcases of the result (d), and (b) is a useful subcase of (c), they are formulated separately in the theorem and proved separately.

Remarks on the Proofs. We make use of the following rules:

(2.22) (a) $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A$ and $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant B \rightsquigarrow \vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A \wedge B$;

(b) $\vdash_{\mathrm{SQ}} A \leqslant \mathcal{C}$ and $\vdash_{\mathrm{SQ}} B \leqslant \mathcal{C} \rightsquigarrow \vdash_{\mathrm{SQ}} A \vee B \leqslant \mathcal{C}$;

(c) $\vdash_{\mathrm{SQ}} \mathcal{C} \sqcap A \leqslant B \leftrightsquigarrow \mathcal{C} \leqslant A \to B$.

which can easily be proved to be admissible within SQ.

(a) (i) we have $\vdash_{\mathrm{SQ}} (A \vee (\neg A \wedge B)) \sqcap \neg A \sqcap \neg B \leqslant \Lambda$, since one can show that $\vdash_{\mathrm{SQ}} \mathrm{V} \leqslant k(A \vee (\neg A \wedge B), \neg A)$, $\neg A \wedge B =_{D_f} (A \vee (\neg A \wedge B)) \wedge \neg A$, $\vdash_{\mathrm{SQ}} \mathrm{V} \leqslant k(\neg A \wedge B, \neg B)$ and $\vdash_{\mathrm{Q}} \neg A \wedge B \wedge \neg B \leqslant \Lambda$. (ii) From $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A \sqcup B$ it follows by means of the rules (SQ3.1) and (SQ3.2) that $\vdash_{\mathrm{SQ}} \mathcal{C} \sqcap \neg A \leqslant \neg B$ and $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A$, or $\vdash_{\mathrm{SQ}} \mathcal{C} \sqcap A \leqslant B$ and $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant B$. Because of (2.22) (c) and (a) we have $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A$ or $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant (A \to B) \wedge B$, and by means of (2.12) (c) we have $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A$ or $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant (\neg A \vee (A \wedge B)) \wedge B$. Since $\vdash_{\mathrm{Q}} \mathrm{V} \leqslant k(\neg A, A \wedge B)$, $\vdash_{\mathrm{Q}} \mathrm{V} \leqslant k(A \wedge B, B)$, one can demonstrate the distributivity $(\neg A \vee (A \wedge B)) \wedge B =_{D_f} (\neg A \wedge B) \vee (A \wedge B)$. Therefore, in both of the cases it follows that $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A \vee (\neg A \wedge B) \vee (A \wedge B)$ and, finally, $\vdash_{\mathrm{SQ}} \mathcal{C} \leqslant A \vee (\neg A \wedge B)$. (iii) Since $\vdash_{\mathrm{Q}} A \leqslant A \vee B$ and $\vdash_{\mathrm{Q}} B \leqslant A \vee B$, it follows by means of (SQ2.1) that $\vdash_{\mathrm{SQ}} A \sqcup B \leqslant A \vee B$ also. (iv) can be proved analogously to (ii).

(b) follows from (a) by means of (2.20).

(c) (i) and (ii) can easily be proved by means of a generalization of the result (b) from a 2-place conjunction to an n-place conjunction. (iii) We show that, if $\vdash_{SQ} \sqcap_{i=1}^{n-1} A_i \leqslant (\sqcap_{i=1}^{n-1} A_i)'$, then $\vdash_{SQ} \sqcap_{i=1}^{n} A_i \leqslant ((\sqcap_{i=1}^{n-1} A_i)' \vee \neg A_n) \wedge A_n$. From (b) (iii) it follows that $\vdash_{SQ} (\sqcap_{i=1}^{n-1} A_i)' \sqcap A_n \leqslant ((\sqcap_{i=1}^{n-1} A_i)' \vee \neg A_n) \wedge A_n$. By means of the inductive hypothesis and (2.18) (e) we obtain $\vdash_{SQ} \sqcap_{i=1}^{n} A_i \leqslant ((\sqcap_{i=1}^{n-1} A_i)' \vee \neg A_n) \wedge A_n$ also. (iv) We show that, if $\vdash_{SQ} \sqcap_{i=1}^{n-1} A_i \leqslant \mathcal{C} \rightharpoonup \vdash_{SQ} (\sqcap_{i=1}^{n-1} A_i)' \leqslant \mathcal{C}$ is valid, then $\vdash_{SQ} \sqcap_{i=1}^{n} A_i \leqslant \mathcal{C} \rightharpoonup \vdash_{SQ} (\sqcap_{i=1}^{n} A_i)' \leqslant \mathcal{C}$ is valid. From $\vdash_{SQ} \sqcap_{i=1}^{n} A_i \leqslant \mathcal{C}$ it follows by means of the inductive hypothesis and (2.18) (e) that $\vdash_{SQ} (\sqcap_{i=1}^{n-1} A_i)' \sqcap A_n \leqslant \mathcal{C}$. From (b) (iv) we obtain $\vdash_{SQ} (\sqcap_{i=1}^{n} A_i)' \leqslant \mathcal{C}$.

(d) Let us consider the figure $\mathcal{C} \leqslant \mathcal{A}$. If $\mathcal{C} \leqslant \mathcal{A}$ is deducible within SQ, it follows from the rules (SQ3.1) and (SQ3.2) that there exists a complete set of predecessors of $\mathcal{C} \leqslant \mathcal{A}$ which, after application of the rules (SQ4) and (2.18) (a), establish $\vdash_{SQ} \mathcal{C} \leqslant \neg(\sqcap_{i=1}^{n} A_i), \vdash_{SQ} \mathcal{C} \leqslant \neg(\sqcap_{i=1}^{m} B_i), \ldots$. From (c) we obtain by means of (2.20) $\vdash_{SQ} \mathcal{C} \leqslant \neg(\sqcap_{i=n}^{1} A_i)', \vdash_{SQ} \mathcal{C} \leqslant \neg(\sqcap_{i=m}^{1} B_i)', \ldots$. By means of (2.22) (a) it follows that $\vdash_{SQ} \mathcal{C} \leqslant \neg(\sqcap_{i=n}^{1} A_i)' \wedge \neg(\sqcap_{i=m}^{1} B_i)' \wedge \ldots$, the conjunction denoted by $\bar{C}_k$. In this way one obtains the conjunctions $\bar{C}_i$ for all possible deductions of figures $\mathcal{C} \leqslant \mathcal{A}$ with $\mathcal{C} \in S$. Now we form the disjunction $\bigvee_i \bar{C}_i$ of all $\bar{C}_i$, and obtain $\vdash_{SQ} \mathcal{C} \leqslant \mathcal{A} \rightharpoonup \vdash_{SQ} \mathcal{C} \leqslant \bigvee_i \bar{C}_i$. On the other hand we have $\vdash_{SQ} \bigvee_i \bar{C}_i \leqslant \mathcal{A}$. This proves (i) and (ii), where the proposition $\bar{A}$ is given by $\bigvee_i \bar{C}_i$. Let us now consider the figure $\mathcal{A} \leqslant \mathcal{C}$. If $\vdash_{SQ} \mathcal{A} \leqslant \mathcal{C}$, it follows from the rules (SQ2.1) and (SQ2.2) that there exists a complete set of predecessors of $\mathcal{A} \leqslant \mathcal{C}$ within a deduction of $\mathcal{A} \leqslant \mathcal{C}$ which, after application of the rules (SQ4) and (2.18) (a), establish $\vdash_{SQ} \sqcap_{i=1}^{n} A_i \leqslant \mathcal{C}, \vdash_{SQ} \sqcap_{i=1}^{m} B_i \leqslant \mathcal{C}, \ldots$. Then it follows from (c) that $\vdash_{SQ} (\sqcap_{i=1}^{n} A_i)' \leqslant \mathcal{C}, \vdash_{SQ} (\sqcap_{i=1}^{m} B_i)' \leqslant \mathcal{C}, \ldots$, and by means of (2.22) (b) that $\vdash_{SQ} (\sqcap_{i=1}^{n} A_i)' \vee (\sqcap_{i=1}^{m} B_i)' \vee \cdots \leqslant \mathcal{C}$. We denote the disjunction by $\tilde{C}_k$. The conjunction of all $\tilde{C}_i$ which can be obtained for all possible deductions of figures $\mathcal{A} \leqslant \mathcal{C}$ satisfies $\vdash_{SQ} \mathcal{A} \leqslant \mathcal{C} \rightharpoonup \vdash_{SQ} \bigwedge_i \tilde{C}_i \leqslant \mathcal{C}$, and $\vdash_{SQ} \mathcal{A} \leqslant \bigwedge_i \tilde{C}_i$. Thus (iii) and (iv) are proved, where $\tilde{A}$ is given by $\bigwedge_i \tilde{C}_i$.

By means of (2.21), a further important property of sequential quantum logic can be proved:

$$(2.23) \quad \vdash_{SQ} \mathcal{A} \leqslant \mathcal{A} \quad \text{and} \quad \mathcal{A}^* =_{D_f} \mathcal{A} \quad \leftrightharpoons \quad \mathcal{A} =_{D_f} A \in L$$

This property establishes a necessary and sufficient condition for a proposition to be formally equivalent to a logical proposition.

Proof. From $\vdash_{SQ} \mathcal{A} \leqslant \mathcal{A}$ it follows by means of (2.21) that there exists a logical proposition $\bar{A} \in L$ such that $\vdash_{SQ} \mathcal{A} \leqslant \bar{A}$ and $\vdash_{SQ} \bar{A} \leqslant \mathcal{A}$. By means of (2.13) (11) we obtain $\mathcal{A} =_{D_f} \mathcal{A} \sqcap \bar{A}$ and $\bar{A} =_{D_f} \bar{A} \sqcap \mathcal{A}$. It follows by means of (2.20) (b) and $\mathcal{A}^* =_{D_f} \mathcal{A}$ that $\bar{A} =_{D_f} \mathcal{A} \sqcap \bar{A}$. Therefore we have $\mathcal{A} =_{D_f} \bar{A}$.

3. ALGEBRAIC REPRESENTATIONS OF QUANTUM LOGIC

After having established the system of sequential quantum logic we now wish to represent this system and particular subsystems by means of algebraic structures. Our point of departure is the construction of the Lindenbaum–Tarski algebra of sequential quantum logic.

3.1. The Lindenbaum–Tarski Algebra of Quantum Logic ASQ

Analogously to the construction of the Lindenbaum–Tarski algebra of classical logic [see, e.g., Bell and Slomson (1974), p. 40], a Lindenbaum–Tarski algebra is also generated by quantum logic.

(3.1) *Definition*. The *Lindenbaum–Tarski algebra of quantum logic* (ASQ) is the structure $\langle E_e, E_L, E_S; \wedge, \vee, \to, k(\ ,\ ), \sqcap, \sqcup, \dashv, \neg, 0, 1\rangle$, where

(a) E_e is the set $\{\{a\}: a \in S_e\}$;

(b) E_L is the set $\{[A]: A \in L\}$ of equivalence classes $[A] := \{B \in L: B =_{D_f} A\}$;

(c) E_S is the set $\{[\mathfrak{A}]: \mathfrak{A} \in S\}$ of equivalence classes $[\mathfrak{A}] := \{\mathfrak{B} \in S: \mathfrak{B} =_{D_f} \mathfrak{A}\}$;

(d) $\wedge$ is the operation: $E_L \times E_L \to E_L$ with $[A] \wedge [B] := [A \wedge B]$;

(e) $\wedge$ is the operation: $E_L \times E_L \to E_L$ with $[A] \vee [B] := [A \vee B]$;

(f) $\to$ is the operation: $E_L \times E_L \to E_L$ with $[A] \to [B] := [A \to B]$;

(g) $k(\ ,\ )$ is the operation: $E_L \times E_L \to E_L$ with $k([A],[B]) := [k(A, B)]$;

(h) $\sqcap$ is the operation: $E_S \times E_S \to E_S$ with $[\mathfrak{A}] \sqcap [\mathfrak{B}] := [\mathfrak{A} \sqcap \mathfrak{B}]$;

(i) $\sqcup$ is the operation: $E_S \times E_S \to E_S$ with $[\mathfrak{A}] \sqcup [\mathfrak{B}] := [\mathfrak{A} \sqcup \mathfrak{B}]$;

(j) $\dashv$ is the operation: $E_S \times E_S \to E_S$ with $[\mathfrak{A}] \dashv [\mathfrak{B}] := [\mathfrak{A} \dashv \mathfrak{B}]$;

(k) $\neg$ is the operation: $E_S \to E_S$ with $\neg[\mathfrak{A}] := [\neg \mathfrak{A}]$;

(l) 0 is the equivalence class $[\Lambda] := \{\mathfrak{A} \in S: \mathfrak{A} =_{D_f} \Lambda\}$;

(m) 1 is the equivalence class $[\mathrm{V}] := \{\mathfrak{A} \in S: \mathfrak{A} =_{D_f} \mathrm{V}\}$.

Remark. Because of the substitution rule (2.13) (m), the operations of the algebra do not depend on the representatives of the equivalence classes. For simplicity we use the same symbols for the operations in the algebra as for the logical and sequential connectives in the object language. The algebra is equipped with a comprehensive set of operations which are generated by the connectives and which are not independent. Since we can reduce the connectives to one 2-place logical connective, one 2-place sequential connective and the 1-place connective $\neg$, and since $\mathrm{V} =_{D_f} \neg \Lambda$, it is sufficient to consider the structure $\langle E_e, E_L, E_S; \wedge, \sqcap, \neg, 0\rangle$ for instance.

Because of (2.18) (g), we can introduce a relation ($\leqslant$) which is defined by

(3.2) $\leqslant$ is a relation $\subseteq E_S \times E_S$ such that

$$[\mathcal{A}] \leqslant [\mathcal{B}] \overset{\text{def.}}{\leftrightsquigarrow} ([\mathcal{A}],[\mathcal{B}]) \in \leqslant \ \leftrightsquigarrow \ \vdash_{SQ} \mathcal{A} \leqslant \mathcal{B}.$$

Since we have $\vdash_{SQ} \mathcal{A} \subseteq \mathcal{B} \ \leftrightsquigarrow \ \mathcal{A} =_{D_f} \mathcal{A} \sqcap \mathcal{B}$, it follows that

$$(3.3) \qquad [\mathcal{A}] \leqslant [\mathcal{B}] \ \leftrightsquigarrow [\mathcal{A}] = [\mathcal{A}] \sqcap [\mathcal{B}]$$

Since in the following we only consider structures of the sets of equivalence classes, we simply use the same symbols for the equivalence classes as for their representatives.

The algebra ASQ, which is defined in (3.1) by means of the relation of formal equivalence of S propositions, will now be formally characterized by the equations which hold between the elements of E_S. Since the equations are generated by the formal equivalences of propositions, they are established by means of translating the scheme (2.13) in the following way: Propositions are replaced by the corresponding equivalence classes. The formal equivalences (2.13) (b)–(k) are replaced by the corresponding equations. (2.13) (a) and (m) are eliminated since they are fulfilled by equations. The rules (2.13) (l1) and (l2), which are reversible, are replaced by the symmetric rules for the corresponding equations, where in the premise the sequential implication is replaced by the relation (3.2). Thus we have the following system for establishing the equations of the algebra ASQ:

(3.4) (a) $\mathcal{A} \sqcap (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{C}$

(b) $\mathcal{A} \sqcap 0 = 0 = 0 \sqcap \mathcal{A}$

(c) $\mathcal{A} \sqcap 1 = \mathcal{A} = 1 \sqcap \mathcal{A}$

(d) $\mathcal{A} \sqcup \mathcal{B} = \neg(\neg \mathcal{A} \sqcap \neg \mathcal{B})$

(e) $\mathcal{A} \dashv \mathcal{B} = \neg \mathcal{A} \sqcup \mathcal{B}$

(f) $\neg(\neg \mathcal{A}) = \mathcal{A}$

(g) $A \wedge B = (A \sqcap B) \sqcap k(A,B)$

(h) $A \vee B = (A \sqcup B) \sqcup \neg k(A,B)$

(i) $A \rightarrow B = A \dashv (B \sqcap k(A,B))$

(j) $k(A,B) = (A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)$

(k1) $\mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \mathcal{A} \leqslant \neg(\neg \mathcal{B} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1)$
$\leftrightsquigarrow \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap \mathcal{A} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1 = \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$

(k2) $\mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \neg \mathcal{A} \leqslant \neg(\mathcal{B} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1)$
$\leftrightsquigarrow \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap \mathcal{B} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1 = \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1$

where A, B stand for elements $\in E_L$, $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ stand for elements $\in E_S$, and $\mathfrak{A}_1, \ldots, \mathfrak{A}_n, \mathfrak{B}_1, \ldots, \mathfrak{B}_n$ stand for elements $\in E_S$ or no element.

Obviously the rule (k1) includes the rule (3.3) as the case in which $\mathfrak{A}_1, \ldots, \mathfrak{A}_n, \mathfrak{B}_1, \ldots, \mathfrak{B}_n$ stand for no element.

In a second step we also wish to classify the relation $\leqslant$ which is used in the rules (k1) and (k2), by means of a scheme. Since the sequential implication generates the relation $\leqslant$, this scheme is obtained by means of the translation of the calculus SQ of sequential quantum logic. At first we consider the subcalculus Q of quantum logic.

3.1.1. Representation of Q by Means of a Free Orthomodular Lattice. The beginnings of the calculus are replaced by the corresponding axioms of the system, i.e., "$\Rightarrow$" is eliminated. The axiom that corresponds to the beginning (Q5.0) is eliminated since it can be obtained by means of (3.3) and (3.4) (c). The constitutive rules of the calculus are replaced by corresponding metalogical inferences, i.e., "$\Rightarrow$" is replaced by "$\rightsquigarrow$." Thus we arrive at the following system which classifies the relation $\leqslant$ with respect to the elements $\in E_L$:

(3.5) (a1) $A \leqslant A$
(a2) $A \leqslant B$ and $B \leqslant C \rightsquigarrow A \leqslant C$
(b1) $A \wedge B \leqslant A$
(b2) $A \wedge B \leqslant B$
(b3) $C \leqslant A$ and $C \leqslant B \rightsquigarrow C \leqslant A \wedge B$
(c1) $A \leqslant A \vee B$
(c2) $B \leqslant A \vee B$
(c3) $A \leqslant C$ and $B \leqslant C \rightsquigarrow A \vee B \leqslant C$
(d1) $A \wedge (A \rightarrow B) \leqslant B$
(d2) $A \wedge C \leqslant B \rightsquigarrow A \rightarrow C \leqslant A \rightarrow B$
(d3) $A \leqslant B \rightarrow A \rightsquigarrow B \leqslant A \rightarrow B$
(d4) $B \leqslant A \rightarrow B$ and $C \leqslant A \rightarrow C \rightsquigarrow B * C \leqslant A \rightarrow (B * C)$
(e1) $A \wedge \neg A \leqslant 0$
(e2) $A \wedge B \leqslant 0 \rightsquigarrow A \rightarrow B \leqslant \neg A$
(e3) $A \leqslant B \rightarrow A \rightsquigarrow \neg A \leqslant B \rightarrow \neg A$
(e4) $1 \leqslant A \vee \neg A$

with $A, B, C \in E_L$ and $* \in \{\wedge, \vee, \rightarrow\}$.

(3.6) *Theorem*. The structure $\langle E_L, \leqslant \rangle$ is a *free orthocomplemented quasimodular* (*orthomodular*) *lattice* generated by the elements of E_e.

Proof. We show at first that $\neg A$ is an *orthocomplement*, i.e.,

(3.7) (a) $A \wedge \neg A \leqslant 0$
(b) $1 \leqslant A \vee \neg A$
(c) $A \leqslant B \rightsquigarrow \neg B \leqslant \neg A$

(a) and (b) are already given by (3.5) (e1) and (e4). In order to prove (c) we make use of (2.12) (c), from which we obtain $A \to B = \neg A \vee (A \wedge B)$. From $A \leqslant B$ it follows by means of (3.5) (a1), (a2), (b3), and (c2) that $A \leqslant \neg B \vee (B \wedge A)$, hence $A \leqslant B \to A$. By means of (3.5) (d3) and (e3) we obtain $\neg B \leqslant A \to \neg B$ and, therefore, $\neg B \leqslant \neg A \vee (A \wedge \neg B)$. From $A \leqslant B$ it follows that $A \wedge \neg B \leqslant 0$; therefore, $\neg B \leqslant \neg A$.

The *quasimodularity*:

$$(3.8) \qquad B \leqslant A \quad \text{and} \quad C \leqslant \neg A \rightsquigarrow A \wedge (B \vee C) \leqslant B$$

can be obtained in the following way. From $B \leqslant A$ it follows by means of (a1), (a2), and (b3) that $B \leqslant A \wedge B$, and by means of (c2) that $B \leqslant \neg A \vee (A \wedge B)$. From $C \leqslant \neg A$ it follows by means of (c1) and (a2) that $C \leqslant \neg A \vee (A \wedge B)$. By means of (c3) we obtain $B \vee C \leqslant \neg A \vee (A \wedge B)$ and therefore $B \vee C \leqslant A \to B$. From this it follows that $A \wedge (B \vee C) \leqslant A \wedge (A \to B)$, and by means of (d.1) that $A \wedge (B \vee C) \leqslant B$.

Because of (3.5) (a1)–(c3) it follows that $\langle E_L, \leqslant \rangle$ is a lattice. On the other hand, the rules (3.5) (d1)–(d4) can be obtained from the rules (3.5) (a1)–(c3) and (3.7) and (3.8). Therefore, the relation $\leqslant$ in $\langle E_L, \leqslant \rangle$ does not contain more elements than those derivable from the postulates for a lattice (3.5) (a1)–(c3), and (3.7), (3.8). This means that $\langle E_L, \leqslant \rangle$ is a *free* orthomodular lattice generated by the elements of E_e.

The equations between the elements of E_L are now given by means of the system (3.5), (3.4) and the restriction of (3.4) (k1) to the elements of E_L:

$$(3.9) \qquad A \leqslant B \leftrightsquigarrow A = A \wedge B$$

which is obtained from $A \leqslant B$ and $B \leqslant A \leftrightsquigarrow A = B$.

With the aid of (3.9) we can also replace the system (3.5) by a system of equations and rules for equations, in order to classify the equations between the elements of E_L.

3.1.2. Representation of SQ by means of a Baer* Semigroup. Analogously to the translation of the calculus Q into the system (3.5), the calculus SQ is transformed into a system, which now establishes all elements of the relation $\leqslant$ defined by (3.2). This system, which comprises the premises of the rules (3.4) (k1) and (k2), together with the system (3.4), classifies all equations between the elements of E_S. However, we obtain a much simpler system for this purpose, if the system and the premises of (3.4) (k1) and (k2) are transformed into equations by means of

$$(3.10) \qquad \mathfrak{A} \leqslant \mathfrak{B} \leftrightsquigarrow \mathfrak{A} \sqcap \neg \mathfrak{B} = 0$$

which is obtained from the translation of (2.18) (a), (SQ4), and (3.4) (k1). Then one can easily show that the translations of the rules (SQ1)–(SQ5) are obtained from the scheme (3.4) where the premises of the rules (k1) and (k2) are transformed according to (3.10). We have

$$A \sqcap \neg B = 0 \rightharpoondown \mathcal{A} \sqcap A \sqcap \neg B = 0$$

by means of (3.4) (b);

$$\mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \mathcal{A} \sqcap \neg \mathcal{B} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1 \sqcap \neg \mathcal{C} = 0$$

and

$$\mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap \mathcal{A} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1 \sqcap \neg \mathcal{C} = 0$$

$$\rightharpoondown \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1 \sqcap \neg \mathcal{C} = 0$$

by means of (3.4) (k1); analogously the translations of (SQ2.2) to (SQ3.2);

$$\mathcal{C} \sqcap \mathcal{A} \sqcap \neg 0 = 0 \rightharpoondown \mathcal{C} \sqcap \neg(\neg \mathcal{A}) = 0$$

by means of (3.5), i.e., $\neg 0 = 1$, (3.4) (c) and (f); the rules that correspond to the rules (SQ5) are immediately obtained from (3.4) (k1) and (k2).

Thus we have obtained the following result: The equations of the algebra of sequential quantum logic ASQ are given by the equations (3.4) (a)–(j), and the rules

$$\begin{aligned}
&(3.4)\quad (\mathrm{k})\quad \mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \mathcal{A} \sqcap \neg \mathcal{B} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1 = 0\\
&\leftrightharpoons \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap \mathcal{A} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1\\
&= \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1\\
&\qquad (\mathrm{l})\quad \mathcal{A}_1 \sqcap \cdots \sqcap \mathcal{A}_n \sqcap \neg \mathcal{A} \sqcap \mathcal{B} \sqcap \mathcal{B}_n \sqcap \cdots \sqcap \mathcal{B}_1 = 0\\
&\leftrightharpoons \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap \mathcal{B} \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1\\
&= \mathcal{A}_1 \sqcap \neg(\mathcal{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathcal{A}_n \sqcap (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{B}_n) \sqcap \cdots) \sqcap \mathcal{B}_2) \sqcap \mathcal{B}_1
\end{aligned}$$

and the system (3.5), together with (3.9).

The properties of sequential quantum logic (2.18)–(2.22) can immediately be transferred to the algebra ASQ. In particular we have because of (2.19) the following:

(3.11) There exists an automorphism $*$: $E_S \rightarrow E_S$, $\mathfrak{A} \mapsto \mathfrak{A}^*$, such that for $\mathfrak{A}, \mathfrak{B} \in E_S$ and $A \in E_L$:

(a) $A^* = A$;
(b) $(\mathfrak{A} \sqcap \mathfrak{B})^* = \mathfrak{B}^* \sqcap \mathfrak{A}^*$;
(c) $(\neg \mathfrak{A})^* = \neg \mathfrak{A}^*$;
(d) $(\mathfrak{A}^*)^* = \mathfrak{A}$.

The set of elements that satisfy $\mathfrak{A} = \mathfrak{A} \sqcap \mathfrak{A} = \mathfrak{A}^*$ is exactly the set E_L.

We can also use (3.11) as a postulate which, in connection with the system (3.4) (a)–(k), (3.5) and (3.9), yields the equations of the algebra ASQ. The rule (3.4) (l) is then obtained by means of (3.11). From (3.4) (k) it follows that

$$\mathfrak{B}_1^* \sqcap \cdots \sqcap \mathfrak{B}_n^* \sqcap \mathfrak{B}^* \sqcap \neg \mathfrak{A}^* \sqcap \mathfrak{A}_n^* \sqcap \cdots \sqcap \mathfrak{A}_1^* = 0$$

$$\leftrightarrows \mathfrak{B}_1^* \sqcap \neg(\mathfrak{B}_2^* \sqcap \neg(\cdots \sqcap \neg(\mathfrak{B}_n^* \sqcap \mathfrak{B}^* \sqcap \mathfrak{A}_n^*) \sqcap \cdots) \sqcap \mathfrak{A}_2^*) \sqcap \mathfrak{A}_1^*$$

$$= \mathfrak{B}_1^* \sqcap \neg(\mathfrak{B}_2^* \sqcap \neg(\cdots \sqcap \neg(\mathfrak{B}_n^* \sqcap (\mathfrak{B}^* \sqcap \mathfrak{A}^*) \sqcap \mathfrak{A}_n^*) \sqcap \cdots) \sqcap \mathfrak{A}_2^*) \sqcap \mathfrak{A}_1^*.$$

If (3.11) is applied on this rule we obtain (3.4) (l).

Because of (2.21) and by means of (3.10) we have the following:

(3.12) There exist two mappings $\bar{}$: $E_S \rightarrow E_S$, $\mathfrak{A} \mapsto \bar{\mathfrak{A}}$, and $\tilde{}$: $E_S \rightarrow E_S$, $\mathfrak{A} \mapsto \tilde{\mathfrak{A}}$, such that for $\mathfrak{A}, \mathfrak{C} \in E_S$ and $\bar{\mathfrak{A}}, \tilde{\mathfrak{A}} \in E_L$:

(a) $\bar{\mathfrak{A}} \sqcap \neg \mathfrak{A} = 0$;
(b) $\mathfrak{C} \sqcap \neg \mathfrak{A} = 0 \rightharpoonup \mathfrak{C} \sqcap \neg \bar{\mathfrak{A}} = 0$;
(c) $\mathfrak{A} \sqcap \neg \tilde{\mathfrak{A}} = 0$;
(d) $\mathfrak{A} \sqcap \neg \mathfrak{C} = 0 \rightharpoonup \tilde{\mathfrak{A}} \sqcap \mathfrak{C} = 0$.

Now we can easily prove the following theorem.

(3.13) *Theorem*. The algebra ASQ is a *Baer * semigroup* (Foulis, 1960) $\langle E_S, \sqcap, *, ' \rangle$, where

(a) $\langle E_S, \sqcap \rangle$ is a *semigroup*, i.e., for $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in E_S$:
$\mathfrak{A} \sqcap (\mathfrak{B} \sqcap \mathfrak{C}) = (\mathfrak{A} \sqcap \mathfrak{B}) \sqcap \mathfrak{C}$.
(b) $\langle E_S, \sqcap, * \rangle$ is an *involution semigroup*, i.e., there exists a mapping $*$: $E_S \rightarrow E_S$, called an *involution*, such that for $\mathfrak{A}, \mathfrak{B} \in E_S$:
(i) $(\mathfrak{A} \sqcap \mathfrak{B})^* = \mathfrak{B}^* \sqcap \mathfrak{A}^*$;
(ii) $(\mathfrak{A}^*)^* = \mathfrak{A}$.
(c) $\langle E_S, \sqcap, * \rangle$ has a zero 0, i.e., for $\mathfrak{A} \in E_S$:
$0 \sqcap \mathfrak{A} = 0 = \mathfrak{A} \sqcap 0$.

(d) There exists a mapping $'$: $E_S \to P(E_S) := \{\mathfrak{A} \in E_S\colon \mathfrak{A} = \mathfrak{A} \sqcap \mathfrak{A} = \mathfrak{A}^*\}$ (called the set of *projections*), $\mathfrak{A} \mapsto \mathfrak{A}'$, such that for $\mathfrak{A} \in E_S$:

$$\{\mathfrak{C} \in E_S\colon \mathfrak{C} \sqcap \mathfrak{A} = 0\} = \{\mathfrak{B} \in E_S\colon \mathfrak{B} = \mathfrak{B} \sqcap \mathfrak{A}'\}.$$

The mapping $'$ is given by the product $^{-} \circ \neg$. By means of (3.12) (b) we have: $\mathfrak{C} \sqcap \mathfrak{A} = 0$, then $\mathfrak{C} \sqcap \neg(\overline{\neg \mathfrak{A}}) = 0$. It follows by means of (3.4) (k) that $\mathfrak{C} = \mathfrak{C} \sqcap (\overline{\neg \mathfrak{A}})$. On the other hand we obtain by means of (3.12) (a) that $\mathfrak{B} \sqcap (\overline{\neg \mathfrak{A}}) \sqcap \mathfrak{A} = 0$. Therefore it follows from $\mathfrak{B} = \mathfrak{B} \sqcap (\overline{\neg \mathfrak{A}})$ that $\mathfrak{B} \sqcap \mathfrak{A} = 0$.

It is a well-known result of the theory of Baer * semigroups (see Foulis, 1960) that, if $\langle E_S, \sqcap, ^*, ' \rangle$ is a Baer * semigroup, $P'(E_S) := \{\mathfrak{A} \in P(E_S)\colon (\mathfrak{A}')' = \mathfrak{A}\}$ (called the set of *closed projections*), and the relation $\leqslant\, \subseteq P'(E_S) \times P'(E_S)$ is defined by $\mathfrak{A} \leqslant \mathfrak{B} \overset{\text{def.}}{\leftrightarrow} \mathfrak{A} = \mathfrak{A} \sqcap \mathfrak{B}$, then the structure $\langle P'(E_S), \leqslant \rangle$ is an orthocomplemented quasimodular lattice. Moreover, the infimum of $\mathfrak{A}$ and $\mathfrak{B}$ ($\inf(\mathfrak{A}, \mathfrak{B})$) is equal to $(\mathfrak{A} \sqcap \mathfrak{B}')' \sqcap \mathfrak{A}$, and the supremum of $\mathfrak{A}$ and $\mathfrak{B}$ ($\sup(\mathfrak{A}, \mathfrak{B})$) is equal to $\inf(\mathfrak{A}', \mathfrak{B}')'$.

It follows for ASQ that $P'(E_S) = \{\mathfrak{A}'\colon \mathfrak{A} \in E_S\} = P(E_S) = E_L$ and, since the partial ordering coincides with (3.9), $\inf(\mathfrak{A}, \mathfrak{B}) = \mathfrak{A} \wedge \mathfrak{B}$, $\sup(\mathfrak{A}, \mathfrak{B}) = \mathfrak{A} \vee \mathfrak{B}$. Therefore $\langle P'(E_S), \leqslant \rangle$ and $\langle E_L, \leqslant \rangle$ coincide.

The result of our investigation of the algebra ASQ can be summarized in the following way: ASQ is a structure which is a Baer * semigroup with respect to the operation $\sqcap$, generated by the sequential conjunction of the logic. However, because of the additional operation $\neg$, generated by the negation of sequential propositions, its structure is richer and can completely be characterized with respect to the equations by the following:

(3.14) ASQ is an algebra $\langle E_s, \sqcap, \neg, ^*, ^- \rangle$, where

(a) $\langle E_S, \sqcap \rangle$ is a semigroup with a zero 0, given by (3.4) (a) and (b), for $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in E_S$:
   (i) $\mathfrak{A} \sqcap (\mathfrak{B} \sqcap \mathfrak{C}) = (\mathfrak{A} \sqcap \mathfrak{B}) \sqcap \mathfrak{C}$;
   (ii) $\mathfrak{A} \sqcap 0 = 0 = 0 \sqcap \mathfrak{A}$.

(b) $\langle E_S, \sqcap, \neg \rangle$ is given by (3.4) (k), i.e., for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}_1, \ldots, \mathfrak{A}_n, \mathfrak{B}_1, \ldots, \mathfrak{B}_n \in E_S$:
   (i) $\mathfrak{A}_1 \sqcap \cdots \sqcap \mathfrak{A}_n \sqcap \mathfrak{A} \sqcap \neg \mathfrak{B} \sqcap \mathfrak{B}_n \sqcap \cdots \sqcap \mathfrak{B}_1 = 0$
   $\leftrightarrow \mathfrak{A}_1 \sqcap \neg(\mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap \mathfrak{A} \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2) \sqcap \mathfrak{B}_1$
   $= \mathfrak{A}_1 \sqcap \neg(\mathfrak{A}_2 \sqcap \neg(\cdots \sqcap \neg(\mathfrak{A}_n \sqcap (\mathfrak{A} \sqcap \mathfrak{B}) \sqcap \mathfrak{B}_n) \sqcap \cdots) \sqcap \mathfrak{B}_2) \sqcap \mathfrak{B}_1$;
   (ii) $\neg(\neg \mathfrak{A}) = \mathfrak{A}$.

(c) $*$ is an automorphism: $E_S \to E_S$, $\mathfrak{A} \mapsto \mathfrak{A}^*$, given by (3.11), i.e., for $\mathfrak{A}, \mathfrak{B} \in E_S$:
(i) $(\mathfrak{A} \sqcap \mathfrak{B})^* = \mathfrak{B}^* \sqcap \mathfrak{A}^*$;
(ii) $(\neg \mathfrak{A})^* = \neg \mathfrak{A}^*$;
(iii) $(\mathfrak{A}^*)^* = \mathfrak{A}$.

(d) $\bar{}$ is a mapping: $E_S \to P(E_S) := \{\mathfrak{A} \in E_S\colon \mathfrak{A} = \mathfrak{A} \sqcap \mathfrak{A} = \mathfrak{A}^*\}$, $\mathfrak{A} \mapsto \bar{\mathfrak{A}}$, given by (3.12) (a), (b), i.e., for $\mathfrak{A}, \mathfrak{C} \in E_S$:
(i) $\bar{\mathfrak{A}} \sqcap \neg \mathfrak{A} = 0$;
(ii) $\mathfrak{C} \sqcap \neg \mathfrak{A} = 0 \rightsquigarrow \mathfrak{C} \sqcap \neg \bar{\mathfrak{A}} = 0$.

Then one can also show that, if $\langle E_S, \sqcap, \neg, {}^*, \bar{}\, \rangle$ is given by (3.14), and the relation $\leqslant \subseteq P(E_S) \times P(E_S)$ is defined by $\mathfrak{A} \leqslant \mathfrak{B} \overset{\text{def.}}{\leftrightsquigarrow} \mathfrak{A} = \mathfrak{A} \sqcap \mathfrak{B}$, then the structure $\langle P(E_S), \leqslant \rangle$ is an orthocomplemented quasimodular lattice with $P(E_S) = \{\bar{\mathfrak{A}}\colon\ \mathfrak{A} \in E_S\}$, $\inf(\mathfrak{A}, \mathfrak{B}) = \overline{\mathfrak{A} \sqcap \mathfrak{B}}$, $\sup(\mathfrak{A}, \mathfrak{B}) = \neg \inf(\neg \mathfrak{A}, \neg \mathfrak{B})$.

3.2. Connection between ASQ and Other "Algebras of Quantum Logic"

Besides orthomodular lattices other algebraic structures have been considered within the framework of the axiomatic approach to quantum mechanical propositional systems. As generalizations of the Boolean algebra of classical logic, the theory of partial Boolean algebras (Kamber, 1964), noncommutative and nonassociative generalizations of Boolean lattices [*Zwerchverbände*, Kröger (1973, 1974)], and an algebra, called *algebra of quantum logic* by Dishkant (1977), are of particular interest. In the following we briefly consider their connection with the algebra ASQ.

(3.15) *Definition*. A *partial Boolean algebra* (PBA) is a structure $\langle E_L, K, \cap, \cup, \neg, 0, 1 \rangle$, where

(a) E_L is a set;

(b) K is a relation $\subseteq E_L \times E_L$, $A \sim B \overset{\text{def.}}{\leftrightsquigarrow} (A, B) \in K$ for $A, B \in E_L$,
$\cap, \cup$ are 2-place operations: $K \to E_L$;
$\neg$ is a 1-place operation: $E_L \to E_L$;
$0, 1$ are 0-place operations $\in E_L$;
with the axioms

(i) K is reflexive and symmetric,
$A \cup A = A$ for $A \in E_L$;

(ii) $0 \sim A$,
$0 \cup A = A \cup 0 = A$ for $A \in E_L$;

(iii) $1 \sim A$,
$1 \cap A = A$ for $A \in E_L$;

(iv) $A \sim B \rightsquigarrow A \sim \neg B$,
$A \cap \neg A = 0$, $A \cup \neg A = 1$ for $A \in E_L$;
(v) $A \sim B$, $A \sim C$, $B \sim C \rightsquigarrow A \sim (B \cap C)$ and $A \sim (B \cup C)$,
$A \sim B$, $A \sim C$, $B \sim C$
$\rightsquigarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Remark. If the 2-place operations are given on $E_L \times E_L$, i.e., $K = E_L \times E_L$, (3.15) becomes a system of axioms for a Boolean algebra.

A partial ordering $\leqslant \subseteq E_L \times E_L$ which is *compatible* (Kamber, 1964) with the partial structure of PBA can be introduced by

(3.16) $A \leqslant B \overset{\text{def.}}{\leftrightsquigarrow} A \sim B$ and $A = A \cap B$.

If PBA satisfies the additional postulate

(3.17) $A \leqslant B$, $B \leqslant C \rightsquigarrow A \sim C$

one can show that the structure $\langle E_L, K, \leqslant \rangle$ is a *semi-Boolean algebra* (Kamber, 1964) given by the following.

(3.18) *Definition.* A *semi-Boolean algebra* (SBA) is a structure $\langle E_L, K, \leqslant, \neg \rangle$, where

(a) E_L is a set; $0, 1 \in E_L$;
(b) $\leqslant$ is a partial ordering $\subseteq E_L \times E_L$ with $0 \leqslant A$, $A \leqslant 1$ for $A \in E_L$;
(c) $\neg$ is an involution; $E_L \to E_L$, $A \mapsto \neg A$, $\neg(\neg A) = A$ with
(i) $A \leqslant B \rightsquigarrow \neg B \leqslant \neg A$;
(ii) $A \leqslant \neg A \rightsquigarrow A = 0$.
(d) K is a relation $\subseteq E_L \times E_L$, called *commensurability relation*, given by
(i) K is symmetric;
(ii) $\leqslant \subseteq K$;
(iii) If $\mathrm{BA} \subseteq E_L$ is a Boolean algebra with respect to $\leqslant$ in E_L, i.e.,
(α) $A \in \mathrm{BA} \rightsquigarrow \neg A \in \mathrm{BA}$,
(β) $A, B \in \mathrm{BA} \rightsquigarrow \inf(A, B)$ exists with respect to $\leqslant$ in E_L and $\inf(A, B) \in \mathrm{BA}$,
then $\mathrm{BA} \times \mathrm{BA} \subseteq K$;
(iv) $S \in E_L$, $S \times S \subseteq K \rightsquigarrow$ there exists a Boolean algebra $\mathrm{BA} \subseteq E_L$ with $S \subseteq \mathrm{BA}$.

Remark. The commensurability relation is unique. The quasimodularity (3.8) holds in SBA.

It follows that, if $A \sim B$, then $A \cap B = \inf(A, B) =: A \wedge B$ and $A \cup B = \sup(A, B) =: A \vee B$ with respect to $\leqslant$ in E_L.

If we demand the additional postulate:

(3.19) For arbitrary $A, B \in E_L$ there exists $A \wedge B$ in E_L, the structure $\langle E_L, K, \leqslant \rangle$ becomes an orthocomplemented quasimodular lattice. The commensurability relation is given by $A \sim B \overset{\text{def.}}{\leftrightarrow} 1 = (A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B) =: k(A, B)$.

On the other hand one can establish a PBA from an orthocomplemented quasimodular lattice $\langle E_L, \leqslant \rangle$ in the following way: The relation $A \sim B \overset{\text{def.}}{\leftrightarrow} 1 = k(A, B)$ is a commensurability relation $K \subseteq E_L \times E_L$. The operations $\sqcap, \sqcup$: $K \to E_L$ are defined by $A \sqcap B = A \wedge B$, $A \sqcup B = A \vee B$. Then the structure $\langle E_L, K, \sqcap, \sqcup, \neg, 0, 1 \rangle$ is a PBA which satisfies (3.17).

A connection between the algebra ASQ and noncommutative and nonassociative generalizations of Boolean lattices can also be established, since the generalized lattice operations are appraised by ASQ.

An algebra, called *geordneter Zwerchverband* (Z), is given by the following.

(3.20) *Definition*. Z is an algebra $\langle E_L, \dot{\wedge}, \dot{\vee} \rangle$, where

(a) E_L is a set;

(b) $\dot{\wedge}, \dot{\vee}$ are operations: $E_L \times E_L \to E_L$, such that for $A, B, C \in E_L$

- (i) $(A \dot{\wedge} B) \dot{\wedge} C = (A \dot{\wedge} B) \dot{\wedge} (B \dot{\wedge} C)$
- (ii) $A \dot{\wedge} A = A$
- (iii) $A \dot{\wedge} (B \dot{\wedge} A) = B \dot{\wedge} A$
- (iv) $(A \dot{\vee} B) \dot{\vee} C = (A \dot{\vee} B) \dot{\vee} (B \dot{\vee} C)$
- (v) $A \dot{\vee} A = A$
- (vi) $A \dot{\vee} (B \dot{\vee} A) = B \dot{\vee} A$
- (vii) $A \dot{\wedge} (B \dot{\vee} A) = A$
- (viii) $A \dot{\vee} (B \dot{\wedge} A) = A$

A partial ordering $\leqslant \subseteq E_L \times E_L$ is defined in Z by

(3.21) $A \leqslant B \overset{\text{def.}}{\leftrightarrow} A = A \dot{\wedge} B$

In this way the algebra Z can be considered to be a generalization of a lattice structure.

The algebra $\langle E_L, \dot{\wedge}, \dot{\vee}, \neg, 0, 1 \rangle$ is obtained from Z by means of the additional axioms:

(3.22) (a) $0, 1 \in E_L$,

(b) $\neg$ is an operation: $E_L \to E_L$, $A \mapsto \neg A$, such that for $A, B, C \in E_L$:

(i) $C \dot{\wedge} A \leqslant B \rightarrow C \leqslant B \dot{\vee} \neg A$

(ii) $\neg A$ is an orthocomplement, i.e.,

$0 = \neg 1, \quad A = A \dot{\wedge} 1, \quad A \dot{\vee} \neg A = 1, \quad A \dot{\wedge} B = \neg(\neg A \dot{\vee} \neg B)$.

This extension of Z can be considered to be a generalization of a Boolean lattice.

Within the framework of sequential quantum logic, the operations $\curlywedge$ and $\curlyvee$ of the algebra Z are appraised in the following way: Let $\langle E_e, E_L, E_S; \wedge, \vee, \rightarrow, k(\ ,\ ), \sqcap, \sqcup, \dashv, \neg, 0, 1\rangle$ be the algebra ASQ of sequential quantum logic. Let $\leqslant \subseteq E_S \times E_S$ be the relation defined by (3.2). Because of (2.21) (b) we have the following.

(3.23) $A \curlywedge B \leqslant A \sqcap B \leqslant (A \vee \neg B) \wedge B$ for $A, B \in E_L$ such that $A \wedge B$ is the greatest element of all elements $\mathcal{C}$ of E_S with respect to $\leqslant$ which satisfy $\mathcal{C} \leqslant A \sqcap B$, and $(A \vee \neg B) \wedge B$ is the least element of all elements $\mathcal{C}$ of E_S with respect to $\leqslant$ which satisfy $A \sqcap B \leqslant \mathcal{C}$.

Let us define the operations $\curlywedge$ and $\curlyvee$: $E_L \times E_L \rightarrow E_L$ by $A \curlywedge B := (A \vee \neg B) \wedge B$, $A \curlyvee B := \neg(\neg A \curlywedge \neg B)$. Then one can show by means of the results of Kröger (1973) that the algebra $\langle E_L, \curlywedge, \curlyvee, \neg, 0, 1\rangle$ is an algebra Z which satisfies the axiom (3.22). The relation $\leqslant$, defined by (3.21), is the relation $\leqslant$ in ASQ restricted to E_L. For the element $B \curlyvee \neg A$, which occurs in the axiom (3.22) (b) (i), we have $B \curlyvee \neg A = A \rightarrow B$.

On the other hand, an algebra Z, which satisfies (3.22), induces an orthocomplemented quasimodular lattice in the following way: Let $\langle E_L, \curlywedge, \curlyvee, \neg, 0, 1\rangle$ be an algebra Z and $\leqslant$ the partial ordering in Z defined by (3.21). Then the structure $\langle E_L, \leqslant\rangle$ is an orthocomplemented quasimodular lattice. The lattice operations are given by $A \wedge B = (A \curlyvee \neg B) \curlywedge B$, $A \vee B = \neg(\neg A \wedge \neg B)$.

Another algebra, which represents a quantum mechanical propositional system, is known from the literature. The algebra, called *algebra of quantum logic* (G), is given by the following definition.

(3.24) *Definition.* G is an algebraic structure $\langle E_L, \cdot, \neg, 0, 1\rangle$, where

(a) E_L is a set, $\cdot$ is an operation $E_L \times E_L \rightarrow E_L$;

(b) $0, 1 \in E_L$ such that for $A \in E_L$:
- (i) $A \cdot 0 = 0 = 0 \cdot A$;
- (ii) $1 = \neg 0$;

(c) $\neg$ is an operation $E_L \rightarrow E_L$, $A \mapsto \neg A$, such that for $A, B \in E_L$:
- (i) $\neg A \cdot A = 0$;
- (i) $B \cdot A = 0 \rightsquigarrow B = B \cdot \neg A$;

(d) the following conditions are satisfied for $A, B, C \in E_L$:
- (i) $(A \cdot B) \cdot C = (A \cdot B) \cdot (B \cdot C)$;
- (ii) $A \cdot (B \cdot C) = (A \cdot C) \cdot (B \cdot C)$;
- (iii) $(\neg(A \cdot B) \cdot B) \cdot A = 0$;
- (iv) $\neg(\neg(A \cdot B) \cdot B) \cdot B = A \cdot B$;
- (v) $A \cdot \neg B = \neg A \cdot B \rightsquigarrow A = B$.

Again, we can introduce a partial ordering $\leqslant \subseteq E_L \times E_L$ in G, which is defined by

(3.25) $A \leqslant B \overset{\text{def.}}{\leftrightsquigarrow} A = A \cdot B$.

The algebra G is closely related to the algebra Z: Let $\langle E_L, \cdot, \neg, 0, 1\rangle$ be an algebra of quantum logic, and $\dot\wedge, \dot\vee$ be operations: $E_L \times E_L \to E_L$ defined by $A \dot\wedge B := A \cdot B$, $A \dot\vee B := \neg(\neg A \dot\wedge \neg B)$. Then the structure $\langle E_L, \dot\wedge, \dot\vee, \neg, 0, 1\rangle$ is an algebra Z which satisfies the axiom (3.22). On the other hand, if $\langle E_L, \dot\wedge, \dot\vee, \neg, 0, 1\rangle$ is an algebra Z and the operation $\cdot$: $E_L \times E_L \to E_L$ is defined by $A \cdot B = A \dot\wedge B$, the structure $\langle E_L, \cdot, \neg, 0, 1\rangle$ is an algebra of quantum logic.

The proof of this can be established by means of the connection between G and an orthomodular lattice structure. In the article by Dishkant (1977) it is shown that: If $\langle E_L, \cdot, \neg, 0, 1\rangle$ is an algebra G and $\leqslant$ the partial ordering defined by (3.25), then the structure $\langle E_L, \leqslant\rangle$ is an orthocomplemented quasimodular lattice, the lattice operations given by $A \wedge B = \neg((\neg A) \cdot B) \cdot B$, $A \vee B = \neg(\neg A \wedge \neg B)$. On the other hand, if $\langle E_L, \leqslant\rangle$ is an orthomodular lattice and the operation $\cdot$: $E_L \times E_L \to E_L$ is defined by $A \cdot B = (A \vee \neg B) \wedge B$, then the structure $\langle E_L, \cdot, \neg, 0, 1\rangle$ is an algebra of quantum logic.

The connection between our algebra ASQ and the structures considered above is summarized in Diagram 5.

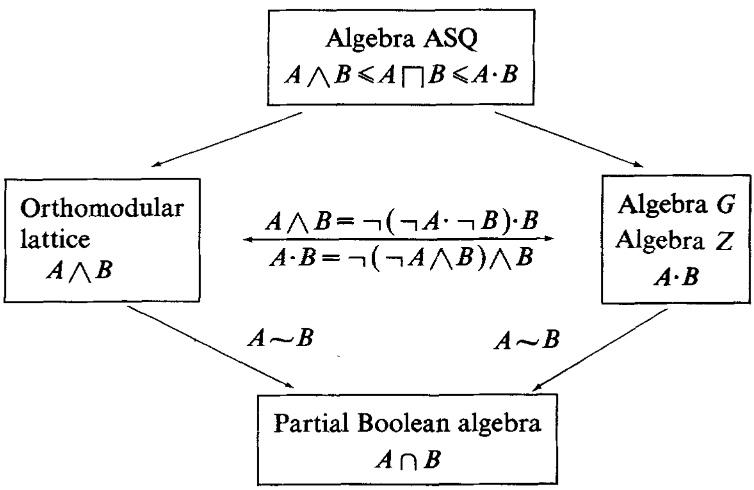


Diagram 5

3.3. Realizations within the Hilbert Space Formalism

It is well known that the lattice of the closed subspaces of a concrete Hilbert space $\mathcal{H}$ is an orthocomplemented quasimodular lattice [see Mittelstaedt (1978), p. 15]. Thus it is a realization of the lattice structure

$\langle E_L, \leqslant \rangle$ given by the axioms (3.5). Thereby the partial ordering $\leqslant$ is the set-theoretic inclusion relation. For any two closed subspaces M_A and M_B there exists a greatest lower bound which is the intersection $M_A \cap M_B$, and there exists a least upper bound which is the closed span $M_A \cup M_B$ of the two subspaces. The orthogonal complement $M_A{}^{\perp}$ of a closed subspace M_A is an orthocomplement of M_A. In order to establish the properties of the lattice, we pass to the algebraic structure of the projection operators which are in one-to-one correspondence with the closed subspaces of $\mathcal{H}$. The lattice $\langle E_L, \leqslant \rangle$ is then realized in the following way:

(3.26) (a) $E_L \rightleftharpoons$ the set $(L_{\mathcal{H}})$ of projection operators $P_A, P_B, \ldots$ on $\mathcal{H}$;
(b) $\leqslant \rightleftharpoons \leqslant \subseteq L_{\mathcal{H}} \times L_{\mathcal{H}}$ such that $P_A \leqslant P_B \overset{\text{def.}}{\leftrightarrow} P_A = P_B \circ P_A$ where $\circ$ denotes the operator multiplication;
(c) $A \wedge B \rightleftharpoons P_{A \wedge B} := (\mathrm{s})\lim_{n\to\infty}(P_A \circ P_B)^n$;
(d) $A \vee B \rightleftharpoons P_{A \vee B} := 1 - (\mathrm{s})\lim_{n\to\infty}((1-P_A)\circ(1-P_B))^n$;
(e) $A \to B \rightleftharpoons P_{A \to B} := 1 - P_A + (\mathrm{s})\lim_{n\to\infty}(P_A \circ P_B)^n$;
(f) $k(A,B) \rightleftharpoons P_{k(A,B)} := P_{A\wedge B} + P_{A \wedge \neg B} + P_{\neg A \wedge B} + P_{\neg A \wedge \neg B}$;
(g) $\neg A \rightleftharpoons 1 - P_A$;
(h) $0 \rightleftharpoons$ zero operator 0;
(i) $1 \rightleftharpoons$ unit operator 1.

It can easily be verified that the relation $\leqslant \subseteq L_{\mathcal{H}} \times L_{\mathcal{H}}$ is a partial ordering. The projection $P_{A \wedge B}$ satisfies the axioms (3.5) (b1)–(b3), and the projection $P_{A \vee B}$ satisfies the axioms (3.5) (c1)–(c3). $P_{A \to B}$ satisfies the axioms (3.5) (d1)–(d4). The elements $k(A,B)$ are realized by the projections $P_{k(A,B)}$. If the elements 0 and 1 are realized by the zero and the unit operators on $\mathcal{H}$, respectively, the projection $P_{\neg A}$ satisfies the axioms (3.5) (e1) to (e4). The projections $P_{A\wedge B}$, $P_{A \vee B}$, $P_{A \to B}$ and $P_{\neg A}$ are uniquely determined by the rules (3.5).

In the particular case that the projections P_A and P_B commute, we have the following:

(3.27) The following equations are equivalent:
(a) $[P_A, P_B]_- = 0$;
(b) $P_{k(A,B)} = 1$;
(c) $P_{A \wedge B} = P_B \circ P_A$;
(d) $P_{A \vee B} = P_A + P_B - P_B \circ P_A$.

In Section 1.3.1 we already used the Hilbert space formalism as an example of establishing the concept of an elementary proposition. Thereby we assumed that elementary propositions are consistently represented by projection operators on a Hilbert space. Elementary propositions, which are considered to be atomic sentences of the object language, must be conceptually distinguished from logically and sequentially connected propositions which are linguistic constructs. When passing to the formal logic in Section 2.2, we stipulated in particular that for any logically connected proposition A there exists an elementary proposition a, such that A and a are value equivalent (i.e., $A =_{D_m} a$). Thus the formal logic of compound propositions provides the elementary propositions with a structure. The structure of elementary propositions established in this way is indeed consistent with the above example of a Hilbert space semantics of elementary propositions, since this structure is realized by the algebra of projection operators on a Hilbert space.

A realization $\langle L_{\mathcal{H}}, \leqslant \rangle$ is *isomorphic* to the subalgebra $\langle E_L, \leqslant \rangle$ of the Lindenbaum–Tarski algebra ASQ, if and only if there exists a one-to-one mapping g from E_L onto $L_{\mathcal{H}}$, such that for $A, B \in E_L$: $A \leqslant B \leftrightarrows g(A) \leqslant g(B)$. However, the algebra $\langle L_{\mathcal{H}}, \leqslant \rangle$ of projection operators on a Hilbert space $\mathcal{H}$ is not isomorphic to $\langle E_L, \leqslant \rangle$. There exist properties of $\langle L_{\mathcal{H}}, \leqslant \rangle$, namely, the *infinite completion* of the lattice operations, and the *atomicity* and *covering property*, which are not properties of $\langle E_L, \leqslant \rangle$. Within the framework of sequential quantum logic, these properties can only be founded by means of an extension of the logic. The infinite completion is obtained by using ∞-place connectives in the language (Denecke, 1977); the atomicity and covering property can be established by means of additional conditions with respect to the language about a physical system. Under such an extension of quantum logic which leads to an extension of its algebraic representation, the existence of an isomorphism between the algebras $\langle L_{\mathcal{H}}, \leqslant \rangle$ and the extension of $\langle E_L, \leqslant \rangle$ can be shown.

Sequentially connected propositions cannot be represented by projection operators in general. However, we show in the following that the set S of sequential propositions can be represented by a more general operator algebra on a Hilbert space which is generated by the set of projection operators and the operations of operator-multiplication and addition.

The system of sequential quantum logic is represented by the algebra ASQ(3.1), the abstract structure of which is given by (3.14). This structure is realized by the following operator algebra (AS) on a Hilbert space:

(3.28) *Definition.* The algebra AS is a structure $\langle L_{\mathcal{H}}, S_{\mathcal{H}}; \sqcap, \neg, {}^*, {}^- \rangle$, where

(a) $L_{\mathcal{H}}$ is the set of projection operators on $\mathcal{H}$.

(b) $S_{\mathcal{H}}$ is the set of operators on $\mathcal{H}$, recursively defined by

(i) $\mathcal{A} \in L_{\mathcal{H}} \rightarrow \mathcal{A} \in S_{\mathcal{H}}$;

(ii) $\mathcal{A}, \mathcal{B} \in S_{\mathcal{H}} \rightarrow \mathcal{A} \circ \mathcal{B},\ 1 - \mathcal{A} \in S_{\mathcal{H}}$.

(c) $\sqcap$ is an operation: $S_{\mathcal{H}} \times S_{\mathcal{H}} \to S_{\mathcal{H}}$, such that for $\mathcal{A}, \mathcal{B} \in S_{\mathcal{H}}$, $\mathcal{A} \sqcap \mathcal{B} := \mathcal{B} \circ \mathcal{A}$.

(d) $\neg$ is an operation: $S_{\mathcal{H}} \to S_{\mathcal{H}}$ such that for $\mathcal{A} \in S_{\mathcal{H}}$, $\neg\mathcal{A} := 1 - \mathcal{A}$.
(e) $*$ is the operator adjoint: $S_{\mathcal{H}} \to S_{\mathcal{H}}$, $\mathcal{A} \mapsto \mathcal{A}^*$.
(f) $^-$ is a mapping: $S_{\mathcal{H}} \to L_{\mathcal{H}}$, which associates each operator $\mathcal{A} \in S_{\mathcal{H}}$ with the projection operator $\bar{\mathcal{A}}$: $\mathcal{H} \to \mathcal{U}_{\mathcal{A}}$, where $\mathcal{U}_{\mathcal{A}} := \{|\varphi\rangle \in \mathcal{H}: \mathcal{A}|\varphi\rangle = |\varphi\rangle\}$ is the *unit space* of $\mathcal{A}$.

(3.29) *Theorem.* The algebra $\langle L_{\mathcal{H}}, S_{\mathcal{H}}; \sqcap, \neg, *, ^-\rangle$ is a realization of the abstract algebra $\langle E_S; \sqcap, \neg, *, ^-\rangle$ given by the axioms (3.14).

Proof. The structure $\langle S_{\mathcal{H}}, \sqcap\rangle$ is a semigroup with a zero, namely, the zero operator 0, since for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in S_{\mathcal{H}}$:
(i) $\mathcal{C}\circ(\mathcal{B}\circ\mathcal{A}) = (\mathcal{C}\circ\mathcal{B})\circ\mathcal{A}$;
(ii) $0\circ\mathcal{A} = 0 = \mathcal{A}\circ 0$.
The operation $\neg$, given by (3.28) (d), satisfies (3.14) (b). (i) If for $\mathcal{A}_1,\ldots,\mathcal{A}_n,\mathcal{A},\mathcal{B},\mathcal{B}_1,\ldots,\mathcal{B}_n \in S_{\mathcal{H}}$: $\mathcal{B}_1\circ\cdots\circ\mathcal{B}_n\circ(1-\mathcal{B})\circ\mathcal{A}\circ\mathcal{A}_n\circ\cdots\circ\mathcal{A}_1 = 0$, then $\mathcal{B}_1\circ\cdots\circ\mathcal{B}_n\circ\mathcal{A}\circ\mathcal{A}_n\circ\cdots\circ\mathcal{A}_1 = \mathcal{B}_1\circ\cdots\circ\mathcal{B}_n\circ\mathcal{B}\circ\mathcal{A}\circ\mathcal{A}_n\circ\cdots\circ\mathcal{A}_1$, and therefore $\mathcal{B}_1\circ(1-\mathcal{B}_2\circ(1-\cdots\circ(1-\mathcal{B}_n\circ\mathcal{A}\circ\mathcal{A}_n)\circ\cdots)\circ\mathcal{A}_2)\circ\mathcal{A}_1 = \mathcal{B}_1\circ(1-\mathcal{B}_2\circ(1-\cdots\circ(1-\mathcal{B}_n\circ\mathcal{B}\circ\mathcal{A}\circ\mathcal{A}_n)\circ\cdots)\circ\cdots)\circ\mathcal{A}_2)\circ\mathcal{A}_1$. Obviously the reverse conclusion is also valid. (ii) $(1-(1-\mathcal{A})) = \mathcal{A}$. The operator adjoint $*$ satisfies the axiom (3.14) (c), since
(i) $(\mathcal{B}\circ\mathcal{A})^* = \mathcal{A}^*\circ\mathcal{B}^*$;
(ii) $(1-\mathcal{A})^* = 1-\mathcal{A}^*$;
(iii) $\mathcal{A}^{**} = \mathcal{A}$.
We have $P(S_{\mathcal{H}}) := \{\mathcal{A} \in S_{\mathcal{H}}: \mathcal{A} = \mathcal{A}\circ\mathcal{A} = \mathcal{A}^*\} = L_{\mathcal{H}}$. The mapping $^-$, defined by (3.28) (f), satisfies the axiom (3.14) (d). We have
(i) $(1-\mathcal{A})\circ\bar{\mathcal{A}} = 0$,
(ii) $(1-\mathcal{A})\circ\mathcal{C} = 0 \rightsquigarrow (1-\bar{\mathcal{A}})\circ\mathcal{C} = 0$,
since $\bar{\mathcal{A}} = \mathcal{A}\circ\bar{\mathcal{A}}$, and $\mathcal{C} = \mathcal{A}\circ\mathcal{C} \rightsquigarrow \mathcal{C} = \bar{\mathcal{A}}\circ\mathcal{C}$. The projection operator $\bar{\mathcal{A}}$ is uniquely determined by (i) and (ii).

The operations $\sqcup$ and $\dashv$ can be defined in $\langle E_S, \sqcap, \neg, *, ^-\rangle$ by $\mathcal{A}\sqcup\mathcal{B} := \neg(\neg\mathcal{A}\sqcap\neg\mathcal{B})$, $\mathcal{A}\dashv\mathcal{B} := \neg\mathcal{A}\sqcup\mathcal{B}$. It follows that they are realized in AS such that:

(3.30) (a) $\mathcal{A}\sqcup\mathcal{B} \rightleftharpoons \mathcal{A}+\mathcal{B}-\mathcal{B}\circ\mathcal{A}$;
(b) $\mathcal{A}\dashv\mathcal{B} \rightleftharpoons 1-\mathcal{A}+\mathcal{B}\circ\mathcal{A}$.

With $P_A, P_B \in L_{\mathcal{H}} = P(S_{\mathcal{H}}) = \{\bar{\mathcal{A}}: \mathcal{A} \in S_{\mathcal{H}}\}$, and with the relation $\leqslant$, defined by (3.26) (b), we have

$$\inf(P_A, P_B) = P_A \circ \overline{(1-(1-P_B)\circ P_A)} = P_{A\wedge B},$$

$$\sup(P_A, P_B) = 1 - \inf(1-P_A, 1-P_B) = P_{A\vee B}.$$

If the algebra $\langle L_{\mathcal{H}}, \leqslant\rangle$ of projection operators on $\mathcal{H}$ is isomorphic to the extension of the substructure $\langle E_L, \leqslant\rangle$ of the Lindenbaum–Tarski algebra ASQ, then the algebra AS is isomorphic to the algebra ASQ. There exists a one-to-one correspondence between the operators in $S_{\mathcal{H}}$ and the equivalence classes of formally equivalent propositions in E_S. The structure of the operations in the algebra corresponds to the connective structure of compound propositions. In this way a *logical foundation* of AS is established.

The set $S_{\mathcal{H}}$, defined by (3.28) (b), is a proper subset of the set $(C_{\mathcal{H}})$ of all continuous linear operators on $\mathcal{H}$. It is exactly the subset which, in the framework of sequential quantum logic, possesses a logical foundation. Whereas, e.g., the operator $P_A + P_B - P_B \circ P_A \in S_{\mathcal{H}}$ corresponds to the sequential disjunction $A \sqcup B$, the operator $P_A + P_B \in C_{\mathcal{H}}$, $\notin S_{\mathcal{H}}$ for $P_A \neq P_{\neg B} \circ P_A$, does not correspond to any proposition composed of the subpropositions A and B.

The algebra (AC) of all continuous linear operators on $\mathcal{H}$ is also a realization of the abstract algebra $\langle E_S, \sqcap, \neg, {}^*, {}^- \rangle$. If the operations are defined according to (3.28) (c)–(f), where $S_{\mathcal{H}}$ is replaced by $C_{\mathcal{H}}$, and $L_{\mathcal{H}}$ is replaced by $P(C_{\mathcal{H}}) := \{\mathcal{Q} \in C_{\mathcal{H}}\colon \mathcal{Q} = \mathcal{Q} \circ \mathcal{Q} = \mathcal{Q}^*\}$, then the proof of theorem (3.29) can be taken over.

We have $P(C_{\mathcal{H}}) = \{\overline{\mathcal{Q}}\colon \mathcal{Q} \in C_{\mathcal{H}}\} = L_{\mathcal{H}}$, and with the relation $\leqslant$, defined by (3.26) (b), we have for $P_A, P_B \in P(C_{\mathcal{H}})$:

$$\inf(P_A, P_B) = P_A \circ \overline{(1-(1-P_B)\circ P_A)} = P_{A \wedge B},$$

$$\sup(P_A, P_B) = 1 - \inf(1-P_A, 1-P_B) = P_{A \vee B}.$$

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